

# HIGHER ORDER SMALLEST PARTS FUNCTIONS AND RANK-CRANK MOMENT INEQUALITIES FROM BAILEY PAIRS

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**ABSTRACT.** We generalize a result of Garvan on inequalities and interpretations of the moments of the partition rank and crank functions. In particular for nearly 30 Bailey pairs, we introduce a rank-like function, establish inequalities with the moments of the rank-like function and an associated crank-like function, and give an associated so called higher order smallest parts function. In some cases we are able to deduce inequalities among the rank-like functions. We also conjecture additional inequalities and a large number of congruences for the higher order smallest parts functions.

## 1. INTRODUCTION

The celebrated Rogers-Ramanujan identities state that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}, \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}},$$

where here and throughout the article

$$(a)_n := (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (a)_{\infty} := (a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^j).$$

While these identities were stated and proved by Rogers in [32], they did not gain attention until they were rediscovered by Ramanujan two decades later. It would be impossible to give here an adequate account of how the Rogers-Ramanujan identities have found their way into various branches of mathematics and related sciences, so we direct the reader to [2, 3, 6, 7, 11, 15, 24, 29]. What we do mention about these identities is that Rogers actually had several more identities of this type, and perhaps most important is that to give uniform proofs of such identities Bailey [13, 14] introduced what would later be known as the Bailey pair machinery. In particular, we recall that a pair of sequences  $(\alpha, \beta)$  form a Bailey pair relative to  $(a, q)$  if

$$\beta_n = \sum_{k=0}^n \frac{\alpha_k}{(q; q)_{n-k} (aq; q)_{n+k}},$$

and a limiting form of Bailey's Lemma is that

$$\sum_{n=0}^{\infty} (x)_n (y)_n \left( \frac{aq}{xy} \right)^n \beta_n = \frac{\left( \frac{aq}{x} \right)_{\infty} \left( \frac{aq}{y} \right)_{\infty}}{(aq)_{\infty} \left( \frac{aq}{xy} \right)_{\infty}} \sum_{n=0}^{\infty} \frac{(x)_n (y)_n \left( \frac{aq}{xy} \right)^n \alpha_n}{\left( \frac{aq}{x} \right)_n \left( \frac{aq}{y} \right)_n}.$$

By letting  $x, y \rightarrow \infty$  in Bailey's Lemma and plugging in the Bailey pair, relative to  $(a, q)$ , given by

$$\beta_n = \frac{1}{(q)_n}, \quad \alpha_n = \frac{(-1)^n (1 - aq^{2n}) (a)_n a^n q^{\frac{n(3n-1)}{2}}}{(1-a)(q)_n},$$

one sees that the right hand side sums to the appropriate products for  $a = 1$  and  $a = q$ , according to the Jacobi triple product identity, so that Bailey has given a uniform proof of both Rogers-Ramanujan identities. After Bailey's success with the above method, Slater [33, 34] demonstrated the incredible power of Bailey pairs by giving over 100 identities of the Rogers-Ramanujan type by further introducing Bailey pairs where

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an appropriate choice of  $x$  and  $y$  would allow the right hand side of Bailey's Lemma to sum to an infinite product.

Besides identities of the Rogers-Ramanujan type, Bailey pairs have found numerous uses in the study of  $q$ -series and integer partitions. We discuss two recent uses related to counting the number of smallest parts in integer partitions. For this we recall that a partition of a positive integer  $n$  is a non-increasing sequence of positive integers that sum to  $n$ ; we agree that there is a single partition of 0, which is the empty partition. We let  $p(n)$  denote the number of partitions of  $n$ . As an example, we see  $p(5) = 7$  as the seven partitions of 5 are given by 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, and 1 + 1 + 1 + 1 + 1. The rank of a partition is given as the largest part minus the number of parts. The crank of a partition is defined as the largest part, if the partition does not contain any ones, and otherwise is the number of parts larger than the number of ones minus the number of ones. Of the partitions of 5 listed previously we see their respective ranks are 4, 2, 1, 0, -1, -2, and -4, whereas their respective cranks are 5, 0, 3, -1, 1, -3, and -5. We let  $N(m, n)$  denote the number of partitions of  $n$  with rank  $m$  and let  $M(m, n)$  denote the number of partitions of  $n$  with crank  $m$ . We call the respective generating functions  $R(z, q)$  and  $C(z, q)$ , that is to say,

$$R(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) z^m q^n, \quad C(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n) z^m q^n.$$

In [5], Andrews introduced the function  $\text{spt}(n)$ , which counts the total number of appearances of the smallest part in each partition of  $n$ . We can think of this as a weighted count on the partitions of  $n$ , where the weight is the number of times the smallest part appears. From the partitions of 5 listed above, we see that  $\text{spt}(5) = 14$ . Given a relation with the second moment of the rank function,  $(z \frac{\partial}{\partial z})^2 R(z, q)|_{z=1}$ , Andrews proved that  $\text{spt}(5n + 4) \equiv 0 \pmod{5}$ ,  $\text{spt}(7n + 5) \equiv 0 \pmod{7}$ , and  $\text{spt}(13n + 6) \equiv 0 \pmod{13}$ . These congruences are reminiscent of Ramanujan's congruences for the partition function  $p(5n + 4) \equiv 0 \pmod{5}$ ,  $p(7n + 5) \equiv 0 \pmod{7}$ , and  $p(11n + 6) \equiv 0 \pmod{11}$ . One can easily verify that a generating function for  $\text{spt}(n)$  is given by

$$S(q) = \sum_{n=0}^{\infty} \text{spt}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)^2 (q^{n+1})_{\infty}}.$$

With this in mind Andrews, Garvan, and Liang [10] introduced a so called  $\text{spt}$ -crank as the series

$$S(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_S(m, n) z^m q^n = \sum_{n=1}^{\infty} \frac{q^n (q^{n+1})_{\infty}}{(z q^n)_{\infty} (z^{-1} q^n)_{\infty}}.$$

It is trivial to see that  $S(1, q) = S(q)$ . Three facts that are not so trivial to prove are that each  $N_S(m, n)$  is a non-negative integer, the 5-dissection of  $S(e^{2\pi i/5}, q)$  gives another proof of the congruence  $\text{spt}(5n + 4) \equiv 0 \pmod{5}$ , and the 7-dissection of  $S(e^{2\pi i/7}, q)$  gives another proof of the congruence  $\text{spt}(7n + 5) \equiv 0 \pmod{7}$ . The essential trick for the dissections is to take the following Bailey pair relative to  $(1, q)$ ,

$$\beta_n = \frac{1}{(q)_n}, \quad \alpha_n = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n (1 + q^n) q^{\frac{n(3n-1)}{2}} & \text{if } n \geq 1, \end{cases}$$

and apply Bailey's Lemma with  $x = z$  and  $y = z^{-1}$  to deduce that

$$(1 - z)(1 - z^{-1})S(z, q) = R(z, q) - C(z, q).$$

Inspired by this identity, Garvan and the second author [22] investigated smallest parts functions related to overpartitions and partitions without repeated odd parts in terms of  $\text{spt}$ -cranks and differences between ranks and cranks. Again the essential trick was to apply Bailey's Lemma with  $x = z$  and  $y = z^{-1}$ , but with a different Bailey pair. In particular, there we used the Bailey pairs

$$\begin{aligned} \beta_n &= \frac{1}{(q^2; q^2)_n}, & \alpha_n &= \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n 2q^{n^2} & \text{if } n \geq 1, \end{cases} \\ \beta_n &= \frac{1}{2} \left( \frac{1}{(q^2; q^2)_n} + \frac{(-1)^n}{(q^2; q^2)_n} \right), & \alpha_n &= \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n (1 + q^{n^2}) & \text{if } n \geq 1, \end{cases} \\ \beta_n &= \frac{1}{(-q, q^2; q^2)_n}, & \alpha_n &= \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n 2q^{2n^2-n} (1 + q^{2n}) & \text{if } n \geq 1, \end{cases} \end{aligned}$$

of which the first two are relative to  $(1, q)$  and the third is relative to  $(1, q^2)$ . Here we note that the same form of Bailey's Lemma gave four identities for spt-crank functions. It is then natural to ask what would happen if one was to look at all of Slater's Bailey pairs in this framework of spt-cranks; Garvan and the second author carried this out in a series of articles [23, 27, 26]. This can be compared to Slater's work, but instead of choosing Bailey pairs that would result in a series that sums to a product by the Jacobi triple product identity, we needed to choose Bailey pairs with  $a = 1$  and such that the associated spt-crank-type function dissected nicely at roots of unity. Altogether this process resulted in over 20 spt-crank-type functions and associated spt-type functions with congruences.

Another use of Bailey's Lemma related to smallest parts functions arose in [21], where Garvan considered the ordinary and symmetrized moments of the rank and crank functions,

$$\begin{aligned} N_k(n) &= \sum_{m=-\infty}^{\infty} m^k N(m, n), & \eta_k(n) &= \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m, n), \\ M_k(n) &= \sum_{m=-\infty}^{\infty} m^k M(m, n), & \mu_k(n) &= \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m, n). \end{aligned}$$

Studies of the ordinary moments of the rank and crank function began with the work of Atkin and Garvan in [12], and Andrews introduced  $\eta_{2k}$ , the symmetrized moment of the rank, in [4]. Previous to Garvan's article, it was conjectured that  $M_{2k}(n) > N_{2k}(n)$  for all positive  $k$  and  $n$  (here only the even moments are of interest as the odd moments are zero). By asymptotics this was known to hold for sufficiently large  $n$  for each  $k$  [19]. Among Garvan's results, we highlight three. The first is the following form of the generating function of  $\mu_{2k}(n) - \eta_{2k}(n)$ ,

$$\sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{\infty} (1 - q^{n_1})^2 \dots (1 - q^{n_k})^2},$$

which clearly exhibits that  $\mu_{2k}(n) \geq \eta_{2k}(n)$  for all  $k$  and  $n$ , and additionally one can easily determine when the inequality is strict. The second is a formula for writing the ordinary moments as a positive integer linear combination of the symmetrized moments, and in particular  $M_{2k}(n) - N_{2k}(n) \geq \mu_2(n) - \eta_2(n)$ . The last is a family of weighted counts of the partitions of  $n$ ,  $\text{spt}_k(n)$ , the higher order smallest parts function, such that the weighting is clearly non-negative and based on the frequency of the parts of the partitions,  $\text{spt}_1(n) = \text{spt}(n)$ , and  $\text{spt}_k(n) = \mu_{2k}(n) - \eta_{2k}(n)$ . Additionally Garvan established a large number of congruences for  $\text{spt}_k(n)$ . The linchpin in establishing the generating function for  $\mu_{2k}(n) - \eta_{2k}(n)$  was again a certain form of Bailey's Lemma applied to the Bailey pair

$$\beta_n = \frac{1}{(q)_n}, \quad \alpha_n = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n (1 + q^n) q^{\frac{n(3n-1)}{2}} & \text{if } n \geq 1. \end{cases}$$

Inspired by Garvan's results, in [25] the second author carried out the same study of differences and inequalities between rank and crank moments related to overpartitions and partitions without repeated odd parts. The upshot was that one needed only choose different Bailey pairs compared to Garvan's work.

It is now obvious what we are to do next. We are to consider the framework developed by Garvan in [21], but applied to all applicable Bailey pairs of Slater. Again this compares with Slater's original use of Bailey pairs. As we will see shortly, the requirements for our choice of Bailey pairs are just that  $a = 1$ ;  $\alpha_0 = \beta_0 = 1$ ; formulaically  $\alpha_{-n} = \alpha_n$ ; and after multiplying by an infinite product, of our own choice, it is clear that  $\beta_n$  has non-negative coefficients. Our results will mirror that of Garvan's study of rank and crank moments. For each Bailey pair considered, we will introduce a rank and crank-like function, obtain a generating function for the difference of symmetrized moments which clearly exhibits non-negative coefficients, deduce an inequality for the associated ordinary moments, and then give a weighted count of partitions that agrees with the difference of the symmetrized moments. To demonstrate that this can be applied to Bailey pairs past those in Slater's list, we also consider one Bailey pair from [16]. Altogether we will give 28 instances of this process.

The rest of the article is organized as follows. In Section 2 we give our definitions and main results, which are series identities and inequalities. In Section 3 we prove the series identities and inequalities listed in Section 2. In Section 4 we prove the combinatorial interpretation of the symmetrized rank and crank moment differences, which justifies our definitions of higher order spt functions. In Section 5 we end with a few conjectures and remarks.

## 2. DEFINITIONS AND STATEMENT OF MAIN RESULTS

For our series identities and inequalities, we need a small number of general identities, all of which are straightforward to prove. These identities have their origins in a combination of classical works on the rank function as well as [4, 21, 25], however here we state and prove them in generality. We combine the main identities into the single following theorem. Given that the proofs primarily already exist in the literature, we will find it takes far longer to state our results than to prove them.

**Theorem 2.1.** *Suppose  $(\alpha, \beta)$  is a Bailey pair relative to  $(1, q)$ ,  $\alpha_0 = \beta_0 = 1$ , and  $\alpha_n = \alpha_{-n}$ . Here we note that by  $\alpha_n = \alpha_{-n}$ , we are treating  $\alpha_n$  as a bilateral sequence by extending  $\alpha_n$  to negative indices according to whatever general formula is given for  $\alpha_n$ . Suppose*

$$R_X(z, q) := P_X(q) \left( 1 + \sum_{n=1}^{\infty} \frac{\alpha_n q^n (1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \right) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_X(m, n) z^m q^n,$$

where  $P_X(q)$  is a series in  $q$ , and for  $k$  a positive integer let

$$N_k^X(n) = \sum_{m=-\infty}^{\infty} m^k N_X(m, n), \quad \eta_k^X(n) = \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N_X(m, n).$$

Then

$$\sum_{n=1}^{\infty} \eta_{2k}^X(n) q^n = -P_X(q) \sum_{n=1}^{\infty} \frac{\alpha_n q^{nk}}{(1-q^n)^{2k}}, \quad (2.1)$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \text{spt}_k^X(n) q^n &:= P_X(q) (q)_{\infty} \sum_{n=1}^{\infty} \mu_{2k}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^X(n) q^n \\ &= P_X(q) \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(q)_{n_1}^2 \beta_{n_1} q^{n_1 + \dots + n_k}}{(1-q^{n_1})^2 \dots (1-q^{n_k})^2}. \end{aligned} \quad (2.2)$$

Furthermore,  $N_k^X(n)$  and  $\eta_k^X(n)$  are zero if  $k$  is odd, and the moments for even  $k$  are related by the identities

$$\eta_{2k}^X(n) = \frac{1}{(2k)!} \sum_{m=-\infty}^{\infty} g_k(m) N_X(m, n), \quad (2.3)$$

$$N_{2k}^X(n) = \sum_{j=1}^k (2j)! S^*(k, j) \eta_{2j}^X(n), \quad (2.4)$$

where  $g_k(x) = \prod_{j=0}^{k-1} (x^2 - j^2)$ , and the sequence  $S^*(k, j)$  is defined recursively by  $S^*(k+1, j) = S^*(k, j-1) + j^2 S^*(k, j)$  and boundary conditions  $S^*(1, 1) = 1$  and  $S^*(k, j) = 0$  if  $j \leq 0$  or  $j > k$ .

**Example.** We first demonstrate the use of this theorem with a specific Bailey pair. Consider the Bailey pair  $B(2)$  of [33],

$$\beta_n = \frac{q^n}{(q)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^n q^{\frac{3n(n-1)}{2}} (1+q^{3n}) & n \geq 1. \end{cases}$$

We note that using this formula with negative  $n$  does give that  $\alpha_n = \alpha_{-n}$ . Looking to Theorem 2.1, if  $\text{spt}_k^X(n)$  is to be a non-negative integer, then we should choose  $P_X(q) = \frac{1}{(q)_{\infty}}$ . From this we now know we should define a rank-like function by

$$R_{B2}(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_{B2}(m, n) z^m q^n = \frac{1}{(q; q)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{n(3n-1)}{2}} (1+q^{3n})}{(1-zq^n)(1-z^{-1}q^n)} \right),$$

and we have the associated ordinary and symmetrized moments given by

$$N_k^{B2}(n) = \sum_{m=-\infty}^{\infty} m^k N_{B2}(m, n), \quad \eta_k^{B2}(n) = \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N_{B2}(m, n).$$

But then

$$\sum_{n=1} (\mu_{2k}(n) - \eta_{2k}^{B2}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1 + n_2 + \dots + n_k}}{(q^{n_1+1})_\infty (1 - q^{n_1})^2 \dots (1 - q^{n_k})^2},$$

and clearly  $\mu_{2k}(n) \geq \eta_{2k}^{B2}(n)$  for positive  $k$  and  $n$ . Additionally by taking  $k = 1$  and examining the  $n_1 = 1$  summand,

$$\frac{q^2}{(q^2)_\infty (1 - q)^2} = q^2 + 2q^3 + 4q^4 + 7q^5 + 12q^6 + 19q^7 + 30q^8 + \dots,$$

we see that  $\mu_2(n) > \eta_2^{B2}(n)$  for  $n \geq 2$ . To obtain an inequality for the ordinary moments, we make the following observation. The  $S^*(k, j)$  are non-negative integers and  $S^*(k, 1)$  is positive for  $k \geq 1$ . In particular, we then have that

$$M_{2k}(n) - N_{2k}^{B2}(n) = \sum_{j=1}^k (2j)! S^*(k, j) (\mu_{2j}(n) - \eta_{2j}^{B2}(n)) \geq \mu_2(n) - \eta_2^{B2}(n).$$

Thus  $M_{2k}(n) > N_{2k}^{B2}(n)$  for all positive  $k$  and  $n \geq 2$ . Furthermore, we can ask what is the non-negative integer  $\text{spt}_k^{B2}(n) := \mu_{2k}(n) - \eta_{2k}^{B2}(n)$  counting in terms of partitions? This is the question we address in Section 4.

We now repeat this process with the many relevant Bailey pairs from [33, 34], and tabulate our results in a corollary. In the cases where a Bailey pair would have fractional powers of  $q$ , we replace  $q$  with the appropriate power of  $q$ . It is worth noting that the Bailey pairs  $B1$ ,  $E1$ , and the unlabeled Bailey pair on page 468 of [33] with  $\beta_n = \frac{1}{(-q^{1/2})_n (q)_n}$ , (which is also  $F1$  with  $q^{1/2} \mapsto -q^{1/2}$ ) correspond respectively to the ordinary rank studied by Garvan in [21] and the Dyson rank for overpartitions and the M2-rank for partitions without repeated odd parts studied by the second author studied in [25]. The M2-rank for overpartitions corresponds to a Bailey pair from a specialization of a finite form of the Jacobi triple product identity. As such, we omit these Bailey pairs from our consideration. In labeling our Bailey pairs, we use the existing label in the literature when it exists, otherwise we label the Bailey pairs relative to  $(1, q)$  as  $X1$ ,  $X2$ ,  $X3$ ,  $X4$ ,  $X5$ ,  $X6$  and the Bailey pairs relative to  $(1, q^2)$  as  $Y1$ ,  $Y2$ ,  $Y3$ ,  $Y4$ . This labeling is not meant to carry any additional semantic value.

To state our corollary, we first introduce the relevant crank-like functions that will appear. In the cases of a crank that has appeared before in the literature, we follow the existing naming conventions. We let

$$\begin{aligned} C(z, q) &= \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n) z^m q^n, \\ \overline{C}(z, q) &= \frac{(-q)_\infty (q)_\infty}{(zq)_\infty (z^{-1}q)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{M}(m, n) z^m q^n, \\ C^J(z, q) &= \frac{(q)_\infty}{(q^3; q^3)_\infty (zq)_\infty (z^{-1}q)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M^J(m, n) z^m q^n, \\ C^{X6}(z, q) &= \frac{(q)_\infty}{(q^2; q^2)_\infty (zq)_\infty (z^{-1}q)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M^{X6}(m, n) z^m q^n, \\ C^F(z, q) &= \frac{(q^2; q^2)_\infty}{(zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M^F(m, n) z^m q^n, \\ C^G(z, q) &= \frac{(-q; q^2)_\infty}{(zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M^G(m, n) z^m q^n, \\ C^Y(z, q) &= \frac{1}{(zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M^Y(m, n) z^m q^n, \\ C^{L2}(z, q) &= \frac{(-q)_\infty (q^4; q^4)_\infty}{(zq^4; q^4)_\infty (z^{-1}q^4; q^4)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M^{L2}(m, n) z^m q^n. \end{aligned}$$

We note that  $C(z, q)$  is the ordinary crank of partitions, the moments of which Garvan studied in [21], and the function  $\overline{C}(z, q)$  is known as the (first residual) crank of overpartitions [17]. Since all of these functions are directly related to the ordinary crank, upon defining the moments in the obvious way, which we omit, based on [21] we have the following,

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu_{2k}(n)q^n &= \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{n(n-1)}{2} + kn} (1 + q^n)}{(1 - q^n)^{2k}}, \\
\sum_{n=1}^{\infty} \bar{\mu}_{2k}(n)q^n &= \frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{n(n-1)}{2} + kn} (1 + q^n)}{(1 - q^n)^{2k}}, \\
\sum_{n=1}^{\infty} \mu_{2k}^J(n)q^n &= \frac{1}{(q^3; q^3)_{\infty} (q)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{n(n-1)}{2} + kn} (1 + q^n)}{(1 - q^n)^{2k}}, \\
\sum_{n=1}^{\infty} \mu_{2k}^{X6}(n)q^n &= \frac{1}{(q)_{\infty} (q^2; q^2)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{n(n-1)}{2} + kn} (1 + q^n)}{(1 - q^n)^{2k}}, \\
\sum_{n=1}^{\infty} \mu_{2k}^F(n)q^n &= \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1) + 2kn} (1 + q^{2n})}{(1 - q^{2n})^{2k}}, \\
\sum_{n=1}^{\infty} \mu_{2k}^G(n)q^n &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1) + 2kn} (1 + q^{2n})}{(1 - q^{2n})^{2k}}, \\
\sum_{n=1}^{\infty} \mu_{2k}^Y(n)q^n &= \frac{1}{(q^2; q^2)_{\infty}^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1) + 2kn} (1 + q^{2n})}{(1 - q^{2n})^{2k}}, \\
\sum_{n=1}^{\infty} \mu_{2k}^{L2}(n)q^n &= \frac{(-q)_{\infty}}{(q^4; q^4)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n(n-1) + 4kn} (1 + q^{4n})}{(1 - q^{4n})^{2k}}.
\end{aligned}$$

We find that the ordinary and symmetrized moments satisfy the same relation as for the rank-like functions. In particular,

$$M_{2k}^X(n) = \sum_{j=1}^k (2j)! S^*(k, j) \mu_{2j}^X(n).$$

We note that if one wishes to actually compute  $\mu_{2k}^X(n) - \eta_{2k}^X(n)$  numerically, one should do so with the above representation for  $\mu_{2k}^X(n)$  and that of  $\eta_{2k}^X(n)$  in (2.1), rather than by (2.2).

**Corollary 2.2.** (1) Using the Bailey pair  $A(1)$  from [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{1}{(q)_{2n}}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ -q^{6k^2 - 5k + 1} & n = 3k - 1, \\ q^{6k^2 - k} + q^{6k^2 + k} & n = 3k, \\ -q^{6k^2 + 5k + 1} & n = 3k + 1, \end{cases}$$

we define

$$\begin{aligned}
R_{A1}(z, q) &= \frac{1}{(q)_{\infty}} \left( 1 - \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{6n^2-2n}}{(1-zq^{3n-1})(1-z^{-1}q^{3n-1})} - \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{6n^2+8n+2}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} \right. \\
&\quad \left. + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{6n^2+2n}(1+q^{2n})}{(1-zq^{3n})(1-z^{-1}q^{3n})} \right),
\end{aligned}$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{A1}(n)q^n = \frac{1}{(q)_{\infty}} \left( \sum_{n=1}^{\infty} \frac{q^{6n^2-5n+1+(3n-1)k}}{(1-q^{3n-1})^{2k}} + \sum_{n=0}^{\infty} \frac{q^{6n^2+5n+1+(3n+1)k}}{(1-q^{3n+1})^{2k}} - \sum_{n=1}^{\infty} \frac{q^{6n^2-n+3nk}(1+q^{2n})}{(1-q^{3n})^{2k}} \right),$$

$$\sum_{n=1}^{\infty} \text{spt}_k^{A1}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{A1}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2}.$$

(2) Using the Bailey pair A(3) from [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{q^n}{(q)_{2n}}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ -q^{6k^2-2k} & n = 3k - 1, \\ q^{6k^2-2k} + q^{6k^2+2k} & n = 3k, \\ -q^{6k^2+2k} & n = 3k + 1, \end{cases}$$

we define

$$R_{A3}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 - \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{6n^2+n-1}}{(1-zq^{3n-1})(1-z^{-1}q^{3n-1})} - \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{6n^2+5n+1}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} \right. \\ \left. + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{6n^2+n}(1+q^{4n})}{(1-zq^{3n})(1-z^{-1}q^{3n})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{A3}(n) q^n = \frac{1}{(q)_{\infty}} \left( \sum_{n=1}^{\infty} \frac{q^{6n^2-2n+(3n-1)k}}{(1-q^{3n-1})^{2k}} + \sum_{n=0}^{\infty} \frac{q^{6n^2+2n+(3n+1)k}}{(1-q^{3n+1})^{2k}} - \sum_{n=1}^{\infty} \frac{q^{6n^2-2n+3nk}(1+q^{4n})}{(1-q^{3n})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{A3}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{A3}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+n_2+\dots+n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2}.$$

(3) Using the Bailey pair A(5) from [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{q^{n^2}}{(q)_{2n}}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ -q^{3k^2-k} & n = 3k - 1, \\ q^{3k^2-k} + q^{3k^2+k} & n = 3k, \\ -q^{3k^2+k} & n = 3k + 1, \end{cases}$$

we define

$$R_{A5}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 - \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{3n^2+2n-1}}{(1-zq^{3n-1})(1-z^{-1}q^{3n-1})} - \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{3n^2+4n+1}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} \right. \\ \left. + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{3n^2+2n}(1+q^{2n})}{(1-zq^{3n})(1-z^{-1}q^{3n})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{A5}(n) q^n = \frac{1}{(q)_{\infty}} \left( \sum_{n=1}^{\infty} \frac{q^{3n^2-n+(3n-1)k}}{(1-q^{3n-1})^{2k}} + \sum_{n=0}^{\infty} \frac{q^{3n^2+n+(3n+1)k}}{(1-q^{3n+1})^{2k}} - \sum_{n=1}^{\infty} \frac{q^{3n^2-n+3nk}(1+q^{2n})}{(1-q^{3n})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{A5}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{A5}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1^2+n_1+n_2+\dots+n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2}.$$

(4) Using the Bailey pair A(7) from [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{q^{n^2-n}}{(q)_{2n}}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ -q^{3k^2-4k+1} & n = 3k - 1, \\ q^{3k^2-2k} + q^{3k^2+2k} & n = 3k, \\ -q^{3k^2+4k+1} & n = 3k + 1, \end{cases}$$

we define

$$R_{A7}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 - \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{3n^2-n}}{(1-zq^{3n-1})(1-z^{-1}q^{3n-1})} - \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{3n^2+7n+2}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} \right)$$

$$+ \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{3n^2+n}(1+q^{4n})}{(1-zq^{3n})(1-z^{-1}q^{3n})},$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{A7}(n) q^n &= \frac{1}{(q)_{\infty}} \left( \sum_{n=1}^{\infty} \frac{q^{3n^2-4n+1+(3n-1)k}}{(1-q^{3n-1})^{2k}} + \sum_{n=0}^{\infty} \frac{q^{3n^2+4n+1+(3n+1)k}}{(1-q^{3n+1})^{2k}} - \sum_{n=1}^{\infty} \frac{q^{3n^2-2n+3nk}(1+q^{4n})}{(1-q^{3n})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{A7}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{A7}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1^2+n_2+\dots+n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}. \end{aligned}$$

(5) Using the Bailey pair  $B(2)$  from [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{q^n}{(q)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^n q^{\frac{3n(n-1)}{2}} (1+q^{3n}) & n \geq 1, \end{cases}$$

we define

$$R_{B2}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{n(3n-1)}{2}} (1+q^{3n})}{(1-zq^n)(1-z^{-1}q^n)} \right),$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{B2}(n) q^n &= \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{3n(n-1)}{2} + nk} (1+q^{3n})}{(1-q^n)^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{B2}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{B2}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+n_2+\dots+n_k}}{(q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}. \end{aligned}$$

(6) Using the Bailey pair  $C(1)$  from [33], which is also  $L(6)$  from [34], relative to  $(1, q)$ ,

$$\beta_n = \frac{1}{(q; q^2)_n (q)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{3k^2-k} (1+q^{2k}) & n = 2k, \\ 0 & n = 2k+1, \end{cases}$$

we define

$$R_{C1}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{3n^2+n} (1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right),$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{C1}(n) q^n &= \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{3n^2-n+2nk} (1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{C1}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{C1}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+\dots+n_k}}{(q; q^2)_{n_1} (q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}. \end{aligned}$$

(7) Using the Bailey pair  $C(2)$  from [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{q^n}{(q; q^2)_n (q)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{3k^2-k} (1+q^{2k}) & n = 2k, \\ (-1)^{k+1} q^{3k^2+k} (1-q^{4k+2}) & n = 2k+1, \end{cases}$$

we define

$$\begin{aligned} R_{C2}(z, q) &= \frac{1}{(q)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{3n^2+n} (1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{n+1} q^{3n^2+3n+1} (1-q^{4n+2})}{(1-zq^{2n+1})(1-z^{-1}q^{2n+1})} \right), \end{aligned}$$



and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{C2}(n) q^n = \frac{1}{(q)_{\infty}} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{3n^2-n+2nk} (1+q^{2n})}{(1-q^{2n})^{2k}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+n+(2n+1)k} (1-q^{4n+2})}{(1-q^{2n+1})^{2k}} \right),$$

$$\sum_{n=1}^{\infty} \text{spt}_k^{C2}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{C2}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+n_2+\dots+n_k}}{(q; q^2)_{n_1} (q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(8) Using the Bailey pair  $C(5)$  from [33], which is also  $L(4)$  from [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{q^{\frac{n(n-1)}{2}}}{(q; q^2)_n (q)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{k^2-k} (1+q^{2k}) & n = 2k, \\ 0 & n = 2k+1, \end{cases}$$

we define

$$R_{C5}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n} (1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{C5}(n) q^n = \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n^2-n+2nk} (1+q^{2n})}{(1-q^{2n})^{2k}},$$

$$\sum_{n=1}^{\infty} \text{spt}_k^{C5}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{C5}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{\frac{n_1(n_1+1)}{2}+n_2+\dots+n_k}}{(q; q^2)_{n_1} (q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(9) Using the Bailey pair  $L(5)$ , upon correcting the formula for  $\beta_n$ , from [34], which also appears as the first entry in the second table of page 468 of [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{(-1)_n}{(q)_n (q; q^2)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ q^{\frac{n(n-1)}{2}} (1+q^n) & n \geq 1, \end{cases}$$

we define

$$R_{L5}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1}) q^{n(n+1)/2} (1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{L5}(n) q^n = \frac{-1}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{\frac{n(n-1)}{2}+nk} (1+q^n)}{(1-q^n)^{2k}},$$

$$\sum_{n=1}^{\infty} \text{spt}_k^{L5}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{L5}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-1)_{n_1} q^{n_1+\dots+n_k}}{(q; q^2)_{n_1} (q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(10) Using the Bailey pair in the seventh entry in the table on page 470 of [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{(-1; q^2)_n}{(q)_{2n}}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{2k^2-k} (1+q^{2k}) & n = 2k, \\ (-1)^k q^{2k^2+k} (1-q^{2k+1}) & n = 2k+1, \end{cases}$$

we define

$$R_{X1}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+n} (1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right. \\ \left. + \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+3n+1} (1-q^{2n+1})}{(1-zq^{2n+1})(1-z^{-1}q^{2n+1})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{X1}(n) q^n = \frac{1}{(q)_{\infty}} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n^2-n+2nk} (1+q^{2n})}{(1-q^{2n})^{2k}} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{2n^2+n+(2n+1)k} (1-q^{2n+1})}{(1-q^{2n+1})^{2k}} \right),$$

$$\sum_{n=1}^{\infty} \text{spt}_k^{X1}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{X1}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-1; q^2)_{n_1} q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(11) Using the Bailey pair in the eighth entry in the table on page 470 of [33], upon correcting the formula for  $\alpha_n$ , relative to  $(1, q)$ ,

$$\beta_n = \frac{q^n (-1; q^2)_n}{(q)_{2n}}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{2k^2-k} (1+q^{2k}) & n = 2k, \\ (-1)^{k+1} q^{2k^2+k} (1-q^{2k+1}) & n = 2k+1, \end{cases}$$

we define

$$R_{X2}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+n} (1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right. \\ \left. + \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{n+1} q^{2n^2+3n+1} (1-q^{2n+1})}{(1-zq^{2n+1})(1-z^{-1}q^{2n+1})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{X2}(n) q^n = \frac{1}{(q)_{\infty}} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n^2-n+2nk} (1+q^{2n})}{(1-q^{2n})^{2k}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+n+(2n+1)k} (1-q^{2n+1})}{(1-q^{2n+1})^{2k}} \right),$$

$$\sum_{n=1}^{\infty} \text{spt}_k^{X2}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{X2}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-1; q^2)_{n_1} q^{2n_1+n_2+\dots+n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(12) Using the Bailey pair in the first entry in the table on page 471 of [33], relative to  $(1, q)$ ,

$$\beta_n = \begin{cases} 1 & n = 0, \\ \frac{(-q^2; q^2)_{n-1}}{(q)_{2n}} & n \geq 1, \end{cases} \quad \alpha_n = \begin{cases} 1 & n = 0, \\ 0 & n = 4k-2, \\ -q^{8k^2-6k+1} & n = 4k-1, \\ q^{8k^2-2k} (1+q^{4k}) & n = 4k, \\ -q^{8k^2+6k+1} & n = 4k+1, \end{cases}$$

we define

$$R_{X3}(z, q) = \frac{1}{(q)_{\infty}} \left( 1 - \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1}) q^{8n^2-2n}}{(1-zq^{4n-1})(1-z^{-1}q^{4n-1})} + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1}) q^{8n^2+2n} (1+q^{4n})}{(1-zq^{4n})(1-z^{-1}q^{4n})} \right. \\ \left. - \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1}) q^{8n^2+10n+2}}{(1-zq^{4n+1})(1-z^{-1}q^{4n+1})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{X3}(n) q^n = \frac{1}{(q)_{\infty}} \left( \sum_{n=1}^{\infty} \frac{q^{8n^2-6n+1+(4n-1)k}}{(1-q^{4n-1})^{2k}} - \sum_{n=1}^{\infty} \frac{q^{8n^2-2n+4nk} (1+q^{4n})}{(1-q^{4n})^{2k}} \right. \\ \left. + \sum_{n=0}^{\infty} \frac{q^{8n^2+6n+1+(4n+1)k}}{(1-q^{4n+1})^{2k}} \right),$$

$$\sum_{n=1}^{\infty} \text{spt}_k^{X3}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{X3}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^2; q^2)_{n_1-1} q^{n_1+\dots+n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(13) Using the Bailey pair in the second entry in the table on page 471 of [33], relative to  $(1, q)$ ,

$$\beta_n = \begin{cases} 1 & n = 0, \\ \frac{q^n(-q^2; q^2)_{n-1}}{(q)_{2n}} & n \geq 1, \end{cases} \quad \alpha_n = \begin{cases} 1 & n = 0, \\ 0 & n = 4k - 2, \\ -q^{8k^2-2k} & n = 4k - 1, \\ q^{8k^2-2k}(1 + q^{4k}) & n = 4k, \\ -q^{8k^2+2k} & n = 4k + 1, \end{cases}$$

we define

$$R_{X4}(z, q) = \frac{1}{(q)_\infty} \left( 1 - \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{8n^2+2n-1}}{(1-zq^{4n-1})(1-z^{-1}q^{4n-1})} + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{8n^2+2}(1+q^{4n})}{(1-zq^{4n})(1-z^{-1}q^{4n})} \right. \\ \left. - \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{8n^2+6n+1}}{(1-zq^{4n+1})(1-z^{-1}q^{4n+1})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{X4}(n) q^n = \frac{1}{(q)_\infty} \left( \sum_{n=1}^{\infty} \frac{q^{8n^2-2n+(4n-1)k}}{(1-q^{4n-1})^{2k}} - \sum_{n=1}^{\infty} \frac{q^{8n^2-2n+4nk}(1+q^{4n})}{(1-q^{4n})^{2k}} + \sum_{n=0}^{\infty} \frac{q^{8n^2+2n+(4n+1)k}}{(1-q^{4n+1})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{X4}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}^{X4}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^2; q^2)_{n_1-1} q^{2n_1+n_2+\dots+n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_\infty (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(14) Using the Bailey pair  $E(4)$  from [33], relative to  $(1, q)$ ,

$$\beta_n = \frac{q^n}{(q^2; q^2)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^n q^{n^2-n}(1+q^{2n}) & n \geq 1, \end{cases}$$

we define

$$R_{E4}(z, q) = \frac{(-q)_\infty}{(q)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2}(1+q^{2n})}{(1-zq^n)(1-z^{-1}q^n)} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{E4}(n) q^n = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n^2-n+kn}(1+q^{2n})}{(1-q^n)^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{E4}(n) q^n := \sum_{n=1}^{\infty} (\bar{\mu}_{2k}(n) - \eta_{2k}^{E4}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^{n_1+1})_\infty q^{2n_1+n_2+\dots+n_k}}{(q^{n_1+1})_\infty (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(15) Using the Bailey pair  $I(14)$  from [34], relative to  $(1, q)$ ,

$$\beta_n = \begin{cases} 1 & n = 0, \\ \frac{(-q^2; q^2)_{n-1}}{(q; q^2)_n (q)_n (-q)_{n-1}} & n \geq 1, \end{cases} \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{2k^2-k}(1+q^{2k}) & n = 2k, \\ 0 & n = 2k + 1, \end{cases}$$

we define

$$R_{I14}(z, q) = \frac{(-q)_\infty}{(q)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+n}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{I14}(n) q^n = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n^2-n+2nk}(1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{I14}(n) q^n := \sum_{n=1}^{\infty} (\bar{\mu}_{2k}(n) - \eta_{2k}^{I14}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^2; q^2)_{n_1-1} (-q^{n_1})_\infty q^{n_1+\dots+n_k}}{(q; q^2)_{n_1} (q^{n_1+1})_\infty (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(16) Using the Bailey pair in Lemma 3.1 from [16], relative to  $(1, q)$ ,

$$\beta_n = \begin{cases} 1 & n = 0, \\ \frac{(-q^3; q^3)_{n-1}}{(-q)_n (q)_{2n-1}} & n \geq 1, \end{cases} \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{\frac{3k(3k-1)}{2}} (1 + q^{3k}) & n = 3k, \\ -2q^{18k^2+9k+1} & n = 6k+1, \\ 2q^{18k^2+15k+3} & n = 6k+2, \\ 2q^{18k^2+21k+6} & n = 6k+4, \\ -2q^{18k^2+27k+10} & n = 6k+5, \end{cases}$$

we define

$$\begin{aligned} R_{X5}(z, q) &= \frac{(-q)_\infty}{(q)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{3n(3n+1)}{2}} (1+q^{3n})}{(1-zq^{3n})(1-z^{-1}q^{3n})} \right. \\ &\quad - 2 \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{18n^2+15n+2}}{(1-zq^{6n+1})(1-z^{-1}q^{6n+1})} + 2 \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{18n^2+21n+5}}{(1-zq^{6n+2})(1-z^{-1}q^{6n+2})} \\ &\quad \left. + 2 \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{18n^2+27n+10}}{(1-zq^{6n+4})(1-z^{-1}q^{6n+4})} - 2 \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{18n^2+33n+15}}{(1-zq^{6n+5})(1-z^{-1}q^{6n+5})} \right), \end{aligned}$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{X5}(n) q^n &= \frac{(-q)_\infty}{(q)_\infty} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{3n(3n-1)}{2} + 3nk} (1+q^{3n})}{(1-q^{3n})^{2k}} + 2 \sum_{n=0}^{\infty} \frac{q^{18n^2+9n+1+(6n+1)k}}{(1-q^{6n+1})^{2k}} \right. \\ &\quad \left. - 2 \sum_{n=0}^{\infty} \frac{q^{18n^2+15n+3+(6n+2)k}}{(1-q^{6n+2})^{2k}} - 2 \sum_{n=0}^{\infty} \frac{q^{18n^2+21n+6+(6n+4)k}}{(1-q^{6n+4})^{2k}} + 2 \sum_{n=0}^{\infty} \frac{q^{18n^2+27n+10+(6n+5)k}}{(1-q^{6n+5})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{X5}(n) q^n &:= \sum_{n=1}^{\infty} (\bar{\mu}_{2k}(n) - \eta_{2k}^{X5}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^3; q^3)_{n_1-1} (-q^{n_1+1})_\infty q^{n_1+\dots+n_k}}{(q^{n_1+1})_{n_1-1} (q^{n_1+1})_\infty (1-q^{n_k})^2 \dots (1-q^{n_1})^2}. \end{aligned}$$

(17) Using the Bailey pair  $J(1)$  from [34], which also appears as equation (3.8) in [33], relative to  $(1, q)$ ,

$$\beta_n = \begin{cases} 1 & n = 0, \\ \frac{(q^3; q^3)_{n-1}}{(q)_{2n-1} (q)_n} & n \geq 1, \end{cases} \quad \alpha_n = \begin{cases} 1 & n = 0, \\ 0 & n = 3k-1, \\ (-1)^k q^{\frac{3k(3k-1)}{2}} (1+q^{3k}) & n = 3k, \\ 0 & n = 3k+1, \end{cases}$$

we define

$$R_{J1}(z, q) = \frac{1}{(q)_\infty (q^3; q^3)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{3n(3n+1)}{2}} (1+q^{3n})}{(1-zq^{3n})(1-z^{-1}q^{3n})} \right),$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{J1}(n) q^n &= \frac{1}{(q)_\infty (q^3; q^3)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{3n(3n-1)}{2} + 3nk} (1+q^{3n})}{(1-q^{3n})^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{J1}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^J(n) - \eta_{2k}^{J1}(n)) q^n \\ &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+\dots+n_k}}{(q)_{2n_1-1} (q^{3n_1}; q^3)_\infty (q^{n_1+1})_\infty (1-q^{n_k})^2 \dots (1-q^{n_1})^2}. \end{aligned}$$

(18) Using the Bailey pair  $J(2)$  from [34], which also appears unlabeled on page 467 of [33], relative to  $(1, q)$ ,

$$\beta_n = \begin{cases} 1 & n = 0, \\ \frac{(q^3; q^3)_{n-1}}{(q)_{2n}(q)_{n-1}} & n \geq 1, \end{cases} \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^{k+1} q^{\frac{9k(k-1)}{2}+1} & n = 3k-1, \\ (-1)^k q^{\frac{3k(3k-1)}{2}} (1+q^{3k}) & n = 3k, \\ (-1)^{k+1} q^{\frac{9k(k+1)}{2}+1} & n = 3k+1, \end{cases}$$

we define

$$R_{J2}(z, q) = \frac{1}{(q)_\infty (q^3; q^3)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{n+1} q^{\frac{3n(3n-1)}{2}}}{(1-zq^{3n-1})(1-z^{-1}q^{3n-1})} \right. \\ \left. + \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{n+1} q^{\frac{3n(3n+5)}{2}+2}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{3n(3n+1)}{2}} (1+q^{3n})}{(1-zq^{3n})(1-z^{-1}q^{3n})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{J2}(n) q^n = \frac{1}{(q)_\infty (q^3; q^3)_\infty} \left( \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{9n(n-1)}{2}+1+(3n-1)k}}{(1-q^{3n-1})^{2k}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{9n(n+1)}{2}+1+(3n+1)k}}{(1-q^{3n+1})^{2k}} \right. \\ \left. - \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n(3n-1)}{2}+3nk} (1+q^{3n})}{(1-q^{3n})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{J2}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}^J(n) - \eta_{2k}^{J2}(n)) q^n \\ = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q)_{n_1-1} (q^{n_1+1})_{n_1} (q^{3n_1}; q^3)_\infty (q^{n_1+1})_\infty (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(19) Using the Bailey pair  $J(3)$  from [34], which also appears unlabeled on page 467 of [33], relative to  $(1, q)$ ,

$$\beta_n = \begin{cases} 1 & n = 0, \\ \frac{q^n (q^3; q^3)_{n-1}}{(q)_{2n} (q)_{n-1}} & n \geq 1, \end{cases} \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^{k+1} q^{\frac{3k(3k-1)}{2}} & n = 3k-1, \\ (-1)^k q^{\frac{3k(3k-1)}{2}} (1+q^{3k}) & n = 3k, \\ (-1)^{k+1} q^{\frac{3k(3k+1)}{2}} & n = 3k+1, \end{cases}$$

we define

$$R_{J3}(z, q) = \frac{1}{(q)_\infty (q^3; q^3)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{n+1} q^{\frac{3n(3n+1)}{2}-1}}{(1-zq^{3n-1})(1-z^{-1}q^{3n-1})} \right. \\ \left. + \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{n+1} q^{\frac{9n(n+1)}{2}+1}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{3n(3n+1)}{2}} (1+q^{3n})}{(1-zq^{3n})(1-z^{-1}q^{3n})} \right),$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{J3}(n) q^n = \frac{1}{(q)_\infty (q^3; q^3)_\infty} \left( \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n(3n-1)}{2}+(3n-1)k}}{(1-q^{3n-1})^{2k}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{3n(3n+1)}{2}+(3n+1)k}}{(1-q^{3n+1})^{2k}} \right. \\ \left. - \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n(3n-1)}{2}+3nk} (1+q^{3n})}{(1-q^{3n})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{J3}(n) q^n := \sum_{n=1}^{\infty} (\mu_{2k}^J(n) - \eta_{2k}^{J3}(n)) q^n \\ = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+n_2+\dots+n_k}}{(q)_{n_1-1} (q^{n_1+1})_{n_1} (q^{3n_1}; q^3)_\infty (q^{n_1+1})_\infty (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

(20) Using the Bailey pair in the ninth entry in the table on page 470 of [33], relative to  $(1, q)$ ,

$$\beta_n = \begin{cases} 1 & n = 0, \\ \frac{(q^2; q^2)_{n-1}}{(q; q^2)_n (q)_n (q)_{n-1}} & n \geq 1, \end{cases} \quad \alpha_n = \begin{cases} 1 & n = 0, \\ q^{2k^2-k}(1+q^{2k}) & n = 2k, \\ 0 & n = 2k+1, \end{cases}$$

we define

$$R_{X6}(z, q) = \frac{1}{(q)_\infty (q^2; q^2)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{2n^2+n}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right),$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{X6}(n) q^n &= \frac{-1}{(q)_\infty (q^2; q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n^2-n+2nk}(1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{X6}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^{X6}(n) - \eta_{2k}^{X6}(n)) q^n \\ &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q)_{n_1-1} (q; q^2)_{n_1} (q^{2n_1}; q^2)_\infty (q^{n_1+1})_\infty (1-q^{n_k})^2 \dots (1-q^{n_1})^2}. \end{aligned}$$

(21) Using the Bailey pair  $F(3)$  from [33], relative to  $(1, q^2)$ ,

$$\beta_n = \frac{1}{q^n (q)_{2n}}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ q^n + q^{-n} & n \geq 1, \end{cases}$$

we define

$$R_{F3}(z, q) = \frac{1}{(q^2; q^2)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^n(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right),$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{F3}(n) q^n &= \frac{-1}{(q^2; q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{2nk-n}(1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{F3}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^{F3}(n) - \eta_{2k}^{F3}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+2n_2+\dots+2n_k}}{(q; q^2)_{n_1} (q^{2n_1+2}; q^2)_\infty (1-q^{2n_k})^2 \dots (1-q^{2n_1})^2}. \end{aligned}$$

(22) Using the Bailey pair  $G(1)$  from [33], which is also  $L(3)$  from [34], relative to  $(1, q^2)$ ,

$$\beta_n = \frac{1}{(-q; q^2)_n (q^4; q^4)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^n q^{\frac{n(3n-1)}{2}} (1+q^n) & n \geq 1, \end{cases}$$

we define

$$R_{G1}(z, q) = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty^2} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{3n(n+1)}{2}} (1+q^n)}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right),$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{G1}(n) q^n &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{n(3n-1)}{2} + 2nk} (1+q^n)}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{G1}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^{G1}(n) - \eta_{2k}^{G1}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^{2n_1+1}; q^2)_\infty q^{2n_1+\dots+2n_k}}{(q^4; q^4)_{n_1} (q^{2n_1+2}; q^2)_\infty^2 (1-q^{2n_k})^2 \dots (1-q^{2n_1})^2}. \end{aligned}$$

(23) Using the Bailey pair  $G(3)$  from [33], relative to  $(1, q^2)$ ,

$$\beta_n = \frac{q^{2n}}{(-q; q^2)_n (q^4; q^4)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^n q^{\frac{3n(n-1)}{2}} (1+q^{3n}) & n \geq 1, \end{cases}$$

we define

$$R_{G3}(z, q) = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty^2} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{n(3n+1)}{2}} (1+q^{3n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right),$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{G3}(n) q^n &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{3n(n-1)}{2} + 2nk} (1+q^{3n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{G3}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^G(n) - \eta_{2k}^{G3}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^{2n_1+1}; q^2)_\infty q^{4n_1+2n_2+\dots+2n_k}}{(q^4; q^4)_{n_1} (q^{2n_1+2}; q^2)_\infty^2 (1-q^{2n_k})^2 \dots (1-q^{2n_1})^2}. \end{aligned}$$

(24) Using the Bailey pair in the first entry in the table on page 470 of [33], relative to  $(1, q^2)$ ,

$$\beta_n = \frac{q^{n^2-2n}}{(q^4; q^4)_n (q; q^2)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{2k^2-3k} (1+q^{6k}) & n = 2k, \\ (-1)^k q^{2k^2-k-1} (1-q^{6k+3}) & n = 2k+1, \end{cases}$$

we define

$$\begin{aligned} R_{Y1}(z, q) &= \frac{1}{(q^2; q^2)_\infty^2} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+n} (1+q^{6n})}{(1-zq^{4n})(1-z^{-1}q^{4n})} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+3n+1} (1-q^{6n+3})}{(1-zq^{4n+2})(1-z^{-1}q^{4n+2})} \right), \end{aligned}$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{Y1}(n) q^n &= \frac{1}{(q^2; q^2)_\infty^2} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n^2-3n+4nk} (1+q^{6n})}{(1-q^{4n})^{2k}} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{2n^2-n-1+(4n+2)k} (1-q^{6n+3})}{(1-q^{4n+2})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{Y1}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^Y(n) - \eta_{2k}^{Y1}(n)) q^n \\ &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1^2+2n_2+\dots+2n_k}}{(q; q^2)_{n_1} (q^4; q^4)_{n_1} (q^{2n_1+2}; q^2)_\infty^2 (1-q^{2n_k})^2 \dots (1-q^{2n_1})^2}. \end{aligned}$$

(25) Using the Bailey pair in the second entry in the table on page 470 of [33], relative to  $(1, q^2)$ ,

$$\beta_n = \frac{q^{n^2}}{(q^4; q^4)_n (q; q^2)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{2k^2-k} (1+q^{2k}) & n = 2k, \\ (-1)^{k+1} q^{2k^2+k} (1-q^{2k+1}) & n = 2k+1, \end{cases}$$

we define

$$\begin{aligned} R_{Y2}(z, q) &= \frac{1}{(q^2; q^2)_\infty^2} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+3n} (1+q^{2n})}{(1-zq^{4n})(1-z^{-1}q^{4n})} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{n+1} q^{2n^2+5n+2} (1-q^{2n+1})}{(1-zq^{4n+2})(1-z^{-1}q^{4n+2})} \right), \end{aligned}$$

and obtain

$$\sum_{n=1}^{\infty} \eta_{2k}^{Y2}(n) q^n = \frac{1}{(q^2; q^2)_\infty^2} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n^2-n+4nk} (1+q^{2n})}{(1-q^{4n})^{2k}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+n+(4n+2)k} (1-q^{2n+1})}{(1-q^{4n+2})^{2k}} \right),$$

$$\begin{aligned}\sum_{n=1}^{\infty} \text{spt}_k^{Y^2}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^Y(n) - \eta_{2k}^{Y^2}(n)) q^n \\ &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1^2 + 2n_1 + 2n_2 + \dots + 2n_k}}{(q; q^2)_{n_1} (q^4; q^4)_{n_1} (q^{2n_1+2}; q^2)_{\infty}^2 (1 - q^{2n_k})^2 \dots (1 - q^{2n_1})^2}.\end{aligned}$$

(26) Using the Bailey pair in the third entry in the table on page 470 of [33], relative to  $(1, q^2)$ ,

$$\beta_n = \frac{1}{(q^4; q^4)_n (q; q^2)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{6k^2 - k} (1 + q^{2k}) & n = 2k, \\ (-1)^k q^{6k^2 + 5k + 1} (1 - q^{2k+1}) & n = 2k + 1, \end{cases}$$

we define

$$\begin{aligned}R_{Y^3}(z, q) &= \frac{1}{(q^2; q^2)_{\infty}^2} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{6n^2+3n} (1+q^{2n})}{(1-zq^{4n})(1-z^{-1}q^{4n})} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{6n^2+9n+3} (1-q^{2n+1})}{(1-zq^{4n+2})(1-z^{-1}q^{4n+2})} \right),\end{aligned}$$

and obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \eta_{2k}^{Y^3}(n) q^n &= \frac{1}{(q^2; q^2)_{\infty}^2} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{6n^2-n+4nk} (1+q^{2n})}{(1-q^{4n})^{2k}} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{6n^2+5n+1+(4n+2)k} (1-q^{2n+1})}{(1-q^{4n+2})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{Y^3}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^Y(n) - \eta_{2k}^{Y^3}(n)) q^n \\ &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1 + \dots + 2n_k}}{(q; q^2)_{n_1} (q^4; q^4)_{n_1} (q^{2n_1+2}; q^2)_{\infty}^2 (1 - q^{2n_k})^2 \dots (1 - q^{2n_1})^2}.\end{aligned}$$

(27) Using the Bailey pair in the fourth entry in the table on page 470 of [33], relative to  $(1, q^2)$ ,

$$\beta_n = \frac{q^{2n}}{(q^4; q^4)_n (q; q^2)_n}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^k q^{6k^2 - 3k} (1 + q^{6k}) & n = 2k, \\ (-1)^{k+1} q^{6k^2 + 3k} (1 - q^{6k+3}) & n = 2k + 1, \end{cases}$$

we define

$$\begin{aligned}R_{Y^4}(z, q) &= \frac{1}{(q^2; q^2)_{\infty}^2} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{6n^2+n} (1+q^{6n})}{(1-zq^{4n})(1-z^{-1}q^{4n})} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{n+1} q^{6n^2+7n+2} (1-q^{6n+3})}{(1-zq^{4n+2})(1-z^{-1}q^{4n+2})} \right),\end{aligned}$$

and obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \eta_{2k}^{Y^4}(n) q^n &= \frac{1}{(q^2; q^2)_{\infty}^2} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{6n^2-3n+4nk} (1+q^{6n})}{(1-q^{4n})^{2k}} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{(-1)^n q^{6n^2+3m+(4n+2)k} (1-q^{6n+3})}{(1-q^{4n+2})^{2k}} \right), \\ \sum_{n=1}^{\infty} \text{spt}_k^{Y^4}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^Y(n) - \eta_{2k}^{Y^4}(n)) q^n\end{aligned}$$



$$= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{4n_1+2n_2+\dots+2n_k}}{(q; q^2)_{n_1} (q^4; q^4)_{n_1} (q^{2n_1+2}; q^2)_\infty^2 (1 - q^{2n_k})^2 \dots (1 - q^{2n_1})^2}.$$

(28) Using the Bailey pair  $L(2)$  from [34], which is also  $M(1)$  from [34], relative to  $(1, q^4)$ ,

$$\beta_n = \frac{(q; q^2)_{2n}}{(q^4; q^4)_{2n}}, \quad \alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^n q^{2n^2-n} (1 + q^{2n}) & n \geq 1, \end{cases}$$

we define

$$R_{L2}(z, q) = \frac{(-q; q)_\infty}{(q^4; q^4)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+3n} (1+q^{2n})}{(1-zq^{4n})(1-z^{-1}q^{4n})} \right),$$

and obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{2k}^{L2}(n) q^n &= \frac{(-q)_\infty}{(q^4; q^4)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n^2-n+4nk} (1+q^{2n})}{(1-q^{4n})^{2k}}, \\ \sum_{n=1}^{\infty} \text{spt}_k^{L2}(n) q^n &:= \sum_{n=1}^{\infty} (\mu_{2k}^{L2}(n) - \eta_{2k}^{L2}(n)) q^n \\ &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{4n_1+\dots+4n_k}}{(q^{4n_1+4}; q^4)_{n_1} (q^{4n_1+1}; q^2)_\infty (q^{4n_1+4}; q^4)_\infty (1 - q^{4n_k})^2 \dots (1 - q^{4n_1})^2}. \end{aligned}$$

As in our example, we see that in each case we have  $\mu_{2k}^X(n) \geq \eta_{2k}^X(n)$ . We note that the symmetrized moments  $\eta_{2k}^X(n)$  for  $X = F3, L5$ , and  $X6$  are non-positive integers;  $\text{spt}_k^{X1}(n) = 2\text{spt}_k^{X3}(n)$ ; and  $\text{spt}_k^{X2}(n) = 2\text{spt}_k^{X4}(n)$ . By determining when  $\mu_2^X(n) > \eta_2^X(n)$ , we also obtain the strict inequalities for  $M_{2k}^X(n) > N_{2k}^X(n)$ , upon noting that

$$M_{2k}^X(n) - N_{2k}^X(n) = \sum_{j=1}^k (2j)! S^*(k, j) (\mu_{2j}^X(n) - \eta_{2j}^X(n)) \geq \mu_2^X(n) - \eta_2^X(n).$$

We record these inequalities in the following table.

TABLE 1. Strict Inequalities for  $M_{2k}^X(n) > N_{2k}^X(n)$ , for positive  $k$ .

$X$	$n$	$X$	$n$	$X$	$n$
$A1$	$n \geq 1$	$X2$	$n \geq 2$	$F3$	$n \geq 1$
$A3$	$n \geq 2$	$X3$	$n \geq 1$	$G1$	$n = 2, n \geq 4$
$A5$	$n \geq 2$	$X4$	$n \geq 1$	$G3$	$n = 4, n \geq 6$
$A7$	$n \geq 1$	$E4$	$n \geq 2$	$Y1$	$n \geq 1$
$B2$	$n \geq 2$	$I14$	$n \geq 2$	$Y2$	$n \geq 3$
$C1$	$n \geq 1$	$X5$	$n \geq 1$	$Y3$	$n \geq 2$
$C2$	$n \geq 2$	$J1$	$n \geq 1$	$Y4$	$n \geq 4$
$C5$	$n \geq 1$	$J2$	$n \geq 1$	$L2$	$n = 4, 8, 9, n \geq 11$
$L5$	$n \geq 1$	$J3$	$n \geq 2$		
$X1$	$n \geq 1$	$X6$	$n \geq 1$		

We can actually determine inequalities between some of the ranks that compare against the same crank. In particular, we see we have additional inequalities for two rank moments  $\eta_{2k}^X(n)$  and  $\eta_{2k}^{X'}(n)$  that are compared against the same crank when

$$\frac{P_X(q) (q)_n^2 \beta_n^X}{(1 - q^n)^2} - \frac{P_{X'}(q) (q)_n^2 \beta_n^{X'}}{(1 - q^n)^2}$$

clearly has non-negative coefficients. We record the identities that yield such results in the follow corollary and omit the identities that would lead to an inequality that is already present. While we could also use these identities to deduce strict inequalities between the relevant ordinary moments, we leave that as an exercise to the interested reader.

**Corollary 2.3.** *For positive  $k$ , we have that*

$$\begin{aligned}
\sum_{n=1}^{\infty} (\eta_{2k}^{B2}(n) - \eta_{2k}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q^{n_1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{B2}(n) - \eta_{2k}^{A3}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1 + n_2 + \dots + n_k}}{(q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2} \times \left( \frac{1}{(q^{n_1+1})_{n_1}} - 1 \right), \\
\sum_{n=1}^{\infty} (\eta_{2k}^{B2}(n) - \eta_{2k}^{C2}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1 + n_2 + \dots + n_k}}{(q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2} \times \left( \frac{1}{(q; q^2)_{n_1}} - 1 \right), \\
\sum_{n=1}^{\infty} (\eta_{2k}(n) - \eta_{2k}^{A1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2} \times \left( \frac{1}{(q^{n_1+1})_{n_1}} - 1 \right), \\
\sum_{n=1}^{\infty} (\eta_{2k}(n) - \eta_{2k}^{C1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2} \times \left( \frac{1}{(q; q^2)_{n_1}} - 1 \right), \\
\sum_{n=1}^{\infty} (\eta_{2k}^{A5}(n) - \eta_{2k}^{A3}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_2})^2} \sum_{j=0}^{n_1-2} q^{jn_1}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{A5}(n) - \eta_{2k}^{A7}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1^2 + n_2 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{A3}(n) - \eta_{2k}^{A1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{A3}(n) - \eta_{2k}^{X4}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1 + n_2 + \dots + n_k} ((-q^2; q^2)_{n_1-1} - 1)}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{A7}(n) - \eta_{2k}^{A1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_2})^2} \sum_{j=0}^{n_1-2} q^{jn_1}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{A1}(n) - \eta_{2k}^{X3}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k} ((-q^2; q^2)_{n_1-1} - 1)}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{X4}(n) - \eta_{2k}^{X3}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^2; q^2)_{n_1-1} q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{X4}(n) - \eta_{2k}^{X2}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^2; q^2)_{n_1-1} q^{2n_1 + n_2 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{C2}(n) - \eta_{2k}^{C1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + n_2 + \dots + n_k}}{(q; q^2)_{n_1} (q^{n_1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{X2}(n) - \eta_{2k}^{X1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-1; q^2)_{n_1} q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{X3}(n) - \eta_{2k}^{X1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^2; q^2)_{n_1-1} q^{n_1 + \dots + n_k}}{(q^{n_1+1})_{n_1} (q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{C5}(n) - \eta_{2k}^{C1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + n_2 + \dots + n_k}}{(q^{n_1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_2})^2} \times \frac{(1 - q^{\frac{n_1(n_1-1)}{2}})}{(q; q^2)_{n_1} (1 - q^{n_1})}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{C1}(n) - \eta_{2k}^{L5}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k} ((-1)_{n_1} - 1)}{(q; q^2)_{n_1} (q^{n_1+1})_{\infty} (1 - q^{n_k})^2 \dots (1 - q^{n_1})^2},
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} (\eta_{2k}^{J3}(n) - \eta_{2k}^{J2}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(q)_{n_1-1} (q^{n_1+1})_{n_1} (q^{3n_1}; q^3)_{\infty} (q^{n_1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{J2}(n) - \eta_{2k}^{J1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+n_2+\dots+n_k}}{(q)_{2n_1} (q^{3n_1}; q^3)_{\infty} (q^{n_1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{E4}(n) - \bar{\eta}_{2k}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^{n_1+1})_{\infty} q^{n_1+\dots+n_k}}{(q^{n_1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_2})^2}, \\
\sum_{n=1}^{\infty} (\bar{\eta}_{2k}(n) - \eta_{2k}^{X5}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^{n_1+1})_{\infty} q^{n_1+\dots+n_k}}{(q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2} \times \left( \frac{(-q^3; q^3)_{n_1-1}}{(q^{n_1+1})_{n_1-1}} - 1 \right), \\
\sum_{n=1}^{\infty} (\bar{\eta}_{2k}(n) - \eta_{2k}^{I14}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^{n_1+1})_{\infty} q^{n_1+\dots+n_k}}{(q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2} \\
&\quad \times \left( \frac{(-q^2; q^2)_{n_1-1} (1+q^{n_1})}{(q; q^2)_{n_1}} - 1 \right), \\
\sum_{n=1}^{\infty} (\eta_{2k}^{G3}(n) - \eta_{2k}^{G1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^{2n_1+1}; q^2)_{\infty} q^{2n_1+\dots+2n_k}}{(q^4; q^4)_{n_1} (q^{2n_1+2}; q^2)_{\infty} (q^{2n_1}; q^2)_{\infty} (1-q^{2n_k})^2 \dots (1-q^{2n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{Y2}(n) - \eta_{2k}^{Y1}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1^2+2n_2+\dots+2n_k}}{(q; q^2)_{n_1} (q^4; q^4)_{n_1} (q^{2n_1+2}; q^2)_{\infty} (q^{2n_1}; q^2)_{\infty}} \\
&\quad \times \frac{1}{(1-q^{2n_k})^2 \dots (1-q^{2n_2})^2}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{Y2}(n) - \eta_{2k}^{Y3}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+\dots+2n_k}}{(q^4; q^4)_{n_1} (q^{2n_1}; q^2)_{\infty} (q^{2n_1+2}; q^2)_{\infty} (1-q^{2n_k})^2 \dots (1-q^{2n_2})^2} \\
&\quad \times \frac{(1-q^{n_1^2})}{(q; q^2)_{n_1} (1-q^{2n_1})}, \\
\sum_{n=1}^{\infty} (\eta_{2k}^{Y4}(n) - \eta_{2k}^{Y3}(n)) q^n &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+\dots+2n_k}}{(q; q^2)_{n_1} (q^4; q^4)_{n_1} (q^{2n_1+2}; q^2)_{\infty} (q^{2n_1}; q^2)_{\infty}} \\
&\quad \times \frac{1}{(1-q^{2n_k})^2 \dots (1-q^{2n_2})^2}.
\end{aligned}$$

Here  $\bar{\eta}_{2k}$  is the symmetrized overpartition rank moment from [25] and satisfies

$$\sum_{n=1}^{\infty} (\bar{\mu}_{2k}(n) - \bar{\eta}_{2k}(n)) q^n = \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(-q^{n_1+1})_{\infty} q^{n_1+\dots+n_k}}{(q^{n_1+1})_{\infty} (1-q^{n_k})^2 \dots (1-q^{n_1})^2}.$$

### 3. THE PROOF OF THEOREM 2.1

**Lemma 3.1.** Suppose  $\alpha_n$  is a sequence such that  $\alpha_n = \alpha_{-n}$ , then

$$\sum_{n=1}^{\infty} \frac{\alpha_n q^n (1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} = \sum_{n \neq 0} \frac{\alpha_n q^n (1-z)}{(1+q^n)(1-zq^n)}.$$

Furthermore, if  $j$  is a positive integer, then

$$\left( \frac{\partial}{\partial z} \right)^j \sum_{n=1}^{\infty} \frac{\alpha_n q^n (1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} = -j! \sum_{n \neq 0} \frac{\alpha_n q^{nj} (1-q^n)}{(1+q^n)(1-zq^n)^{j+1}}.$$

*Proof.* We note the first identity is the standard rearrangements used for many rank functions. We have that

$$\sum_{n=1}^{\infty} \frac{\alpha_n q^n (1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} = \sum_{n=1}^{\infty} \frac{\alpha_n q^n}{1+q^n} \left( \frac{1-z}{1-zq^n} + \frac{1-z^{-1}}{1-z^{-1}q^n} \right)$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{\alpha_n q^n (1-z)}{(1+q^n)(1-zq^n)} + \sum_{n=-\infty}^{-1} \frac{\alpha_{-n} q^{-n} (1-z^{-1})}{(1+q^{-n})(1-z^{-1}q^{-n})} \\
&= \sum_{n=1}^{\infty} \frac{\alpha_n q^n (1-z)}{(1+q^n)(1-zq^n)} + \sum_{n=-\infty}^{-1} \frac{\alpha_n q^n (1-z)}{(1+q^n)(1-zq^n)} \\
&= \sum_{n \neq 0} \frac{\alpha_n q^n (1-z)}{(1+q^n)(1-zq^n)}.
\end{aligned}$$

This establishes the first identity. The second identity follows from the first, upon noting that

$$\left( \frac{\partial}{\partial z} \right)^j \frac{1-z}{1-zq^n} = \frac{-j!(1-q^n)q^{n(j-1)}}{(1-zq^n)^{j+1}}.$$

□

**Lemma 3.2.** Suppose  $\alpha_n$  is a sequence such that  $\alpha_n = \alpha_{-n}$  and

$$R_X(z, q) := P_X(q) \left( 1 + \sum_{n=1}^{\infty} \frac{\alpha_n q^n (1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \right) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_X(m, n) z^m q^n,$$

where  $P_X(q)$  is some series in  $q$ . Let

$$\eta_k^X(n) = \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N_X(m, n).$$

Then  $\eta_{2k+1}^X(n) = 0$  and

$$\sum_{n=1}^{\infty} \eta_{2k}^X(n) q^n = -P_X(q) \sum_{n=1}^{\infty} \frac{\alpha_n q^{nk}}{(1-q^n)^{2k}}.$$

*Proof.* We note this is the generalization of Theorems 1 and 2 of [4]. The proof that  $\eta_{2k+1}^X = 0$  follows that of Theorem 1 of [4] verbatim, as we have  $N_X(m, n) = N_X(-m, n)$  due to the symmetry in  $z$  and  $z^{-1}$ . For the even moments, much the same as in the proof of Theorem 2 from [5], by Lemma 3.1 we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \eta_{2k}^X(n) q^n &= \frac{1}{(2k)!} \left( \frac{\partial}{\partial z} \right)^{2k} z^{k-1} R_X(z, q) \Big|_{z=1} \\
&= \frac{1}{(2k)!} \sum_{j=0}^{k-1} \binom{2k}{j} (k-1) \cdots (k-j) R_X^{(2k-j)}(1, q) \\
&= -P_X(q) \sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{n \neq 0} \frac{\alpha_n q^{n(2k-j)} (1-q^n)}{(1+q^n)(1-q^n)^{2k-j+1}} \\
&= -P_X(q) \sum_{n \neq 0} \frac{\alpha_n q^{2nk}}{(1-q^n)^{2k}(1+q^n)} \sum_{j=0}^{k-1} \binom{k-1}{j} (q^{-n}(1-q^n))^j \\
&= -P_X(q) \sum_{n \neq 0} \frac{\alpha_n q^{2nk}}{(1-q^n)^{2k}(1+q^n)} (1+q^{-n}(1-q^n))^{k-1} \\
&= -P_X(q) \sum_{n \neq 0} \frac{\alpha_n q^{n+nk}}{(1-q^n)^{2k}(1+q^n)} \\
&= -P_X(q) \sum_{n=1}^{\infty} \frac{\alpha_n q^{nk}}{(1-q^n)^{2k}}.
\end{aligned}$$

□

With the above lemma, we have established (2.1) of Theorem 2.1. We then find (2.2) follows immediately from equation (3.3) and Theorem 3.3 of [21], both of which we state for completeness. Equation (3.3) is that for positive  $k$ ,

$$\sum_{n=1}^{\infty} \mu_{2k}(n) q^n = \frac{1}{(q)_{\infty}} \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(1 - q^{n_k})^2 \dots (1 - q^{n_1})^2}.$$

Theorem 3.3 states that if  $\alpha_n$  and  $\beta_n$  are a Bailey pair relative to  $(1, q)$  and  $\alpha_0 = \beta_0 = 1$ . Then

$$\begin{aligned} & \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{(q)_{n_1}^2 q^{n_1 + \dots + n_k} \beta_{n_1}}{(1 - q^{n_k})^2 (1 - q^{n_{k-1}})^2 \dots (1 - q^{n_1})^2} \\ &= \sum_{n_k \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + \dots + n_k}}{(1 - q^{n_k})^2 (1 - q^{n_{k-1}})^2 \dots (1 - q^{n_1})^2} + \sum_{r=1}^{\infty} \frac{q^{kr} \alpha_r}{(1 - q^r)^{2k}}. \end{aligned}$$

Lastly, for (2.3) and (2.4), we find that one can follow verbatim the proof of Theorem 4.3 of [21], as the only requirement is that  $N_X(m, n) = N_X(-m, n)$ . Thus Theorem 2.1 follows.

#### 4. COMBINATORIAL INTERPRETATIONS

To determine what the functions  $\text{spt}_{2k}^X(n)$  are counting, we use the weight from [21] as extended to vector partitions in [25]. We recall that a vector partition of  $n$  is a vector of partitions,  $(\pi_1, \dots, \pi_r)$ , such that altogether the parts sum to  $n$ . When  $\vec{\pi}$  is a vector partition of  $n$ , we write  $|\vec{\pi}| = n$ . For a partition  $\pi$  with different parts  $n_1 < n_2 < \dots < n_m$ , we take  $f_j(\pi)$  to be the frequency of the part  $n_j$ . Given a vector partition  $\vec{\pi} = (\pi_1, \pi_2, \dots, \pi_r)$ , we let  $f_j^1 := f_j^1(\vec{\pi}) = f_j(\pi_1)$ , and have the weight  $\omega_k(\vec{\pi})$  given by

$$\omega_k(\vec{\pi}) := \sum_{\substack{m_1 + m_2 + \dots + m_r = k \\ 1 \leq r \leq k}} \binom{f_1^1 + m_1 - 1}{2m_1 - 1} \sum_{2 \leq j_2 < j_3 < \dots < j_r} \binom{f_{j_2}^1 + m_2}{2m_2} \binom{f_{j_3}^1 + m_3}{2m_3} \dots \binom{f_{j_r}^1 + m_r}{2m_r}.$$

We note that the first sum is over all compositions of  $k$ . Based on the combinatorial interpretations in [21, 25], we expect to find that  $\text{spt}_{2k}^X(n)$  is the number of vector partitions  $\vec{\pi} = (\pi_1, \pi_2, \dots, \pi_r)$ , weighted by  $\omega_k$ , where  $(\pi_2, \dots, \pi_r)$  are restricted to the vector partitions enumerated by  $(q^{n_1+1})_{\infty} (q)_{n_1}^2 P_X(q) \beta_{n_1}$ , where  $n_1$  is the smallest part of  $\pi_1$ . Upon minor adjustments for the cases where  $q \rightarrow q^2$  and  $q \rightarrow q^4$ , this is correct. In a small number of cases one of the partitions  $\pi_k$ , for  $k \geq 2$ , strictly speaking may not be a partition in that we will allow a non-positive part to appear.

**Definition 4.1.** We define the follows sets of vector partitions. In all cases we require that  $\pi_1$  be non-empty.

- $S^{A1}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where  $\pi_1$  has smallest part  $n$  and the parts  $m$  of  $\pi_2$  satisfy  $n + 1 \leq m \leq 2n$ .
- $S^{A3}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $n \leq m \leq 2n$ ; and the part  $n$  appears exactly once in  $\pi_2$ .
- $S^{A5}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $n \leq m \leq 2n$ ; and the part  $n$  appears exactly  $n$  times in  $\pi_2$ .
- $S^{A7}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $n \leq m \leq 2n$ ; and the part  $n$  appears exactly  $n - 1$  times in  $\pi_2$ .
- $S^{B2}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where  $\pi_1$  has smallest part  $n$  and  $\pi_2$  consists of exactly one copy of the part  $n$ .
- $S^{C1}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where the smallest part of  $\pi_1$  is  $n$  and the parts  $m$  of  $\pi_2$  are odd and satisfy  $1 \leq m \leq 2n - 1$ .
- $S^{C2}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where the smallest part of  $\pi_1$  is  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $1 \leq m \leq 2n - 1$ ; the parts of  $\pi_2$  other than  $n$  are odd; if  $n$  is odd, then  $n$  appears at least once in  $\pi_2$ ; and if  $n$  is even, then  $n$  appears exactly once in  $\pi_2$ .
- $S^{C5}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where the smallest part of  $\pi_1$  is  $n$ ; the parts  $m$  of  $\pi_2$  are odd and satisfy  $1 \leq m \leq 2n - 1$ ; and  $\pi_3$  consists of exactly one copy of the part  $\frac{n(n-1)}{2}$ .
- $S^{L5}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  are odd and satisfy  $1 \leq m \leq 2n - 1$ ; and the parts  $m$  of  $\pi_3$  are distinct and satisfy  $0 \leq m \leq n - 1$  (so that  $\pi_3$  may contain 0 at most once).

- $S^{X1}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $n+1 \leq m \leq 2n$ ; and the parts  $m$  of  $\pi_3$  are distinct even parts and satisfy  $0 \leq m \leq 2n-2$  (so that  $\pi_3$  may contain the part 0 at most once).
- $S^{X2}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $n \leq m \leq 2n$ ; the part  $n$  appears exactly once in  $\pi_2$ ; and the parts  $m$  of  $\pi_3$  are distinct even parts and satisfy  $0 \leq m \leq 2n-2$  (so that  $\pi_3$  may contain the part 0 at most once).
- $S^{X3}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where  $\pi_1$  has smallest  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $n+1 \leq m \leq 2n$ ; and the parts  $m$  of  $\pi_3$  are distinct even parts and satisfy  $1 \leq m \leq 2n-2$ .
- $S^{X4}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where  $\pi_1$  has smallest  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $n \leq m \leq 2n$ ; the part  $n$  appears exactly once in  $\pi_2$ ; and the parts  $m$  of  $\pi_3$  are distinct even parts and satisfy  $1 \leq m \leq 2n-2$ .
- $S^{E4}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  are distinct and satisfy  $n \leq m$ ; and the part  $n$  appears exactly once in  $\pi_2$ .
- $S^{I14}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $1 \leq m \leq 2n-1$ ; the even parts of  $\pi_2$  are distinct; and the parts  $m$  of  $\pi_3$  are distinct and satisfy  $n \leq m$ .
- $S^{X5}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3, \pi_4)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  are distinct multiples of 3 and satisfy  $1 \leq m \leq 3n-3$ ; the parts  $m$  of  $\pi_3$  are distinct and satisfy  $n+1 \leq m$ ; and the parts  $m$  of  $\pi_4$  satisfy  $n+1 \leq m \leq 2n-1$ .
- $S^{J1}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where  $\pi_1$  has smallest part  $n$ ;  $\pi_2$  has no part  $m$  such that  $2n \leq m \leq 3n-1$ ; and the parts  $m$  of  $\pi_2$  satisfying  $3n \leq m$  are multiples of 3.
- $S^{J2}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where  $\pi_1$  has smallest part  $n$ ;  $\pi_2$  has no part  $m$  such that  $m = n$  or  $2n+1 \leq m \leq 3n-1$ ; and the parts  $m$  of  $\pi_2$  satisfying  $3n \leq m$  are multiples of 3.
- $S^{J3}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where  $\pi_1$  has smallest part  $n$ ;  $\pi_2$  has no part  $m$  such that  $2n+1 \leq m \leq 3n-1$ ; the part  $n$  appears exactly once in  $\pi_2$ ; and the parts  $m$  of  $\pi_2$  satisfying  $3n \leq m$  are multiples of 3.
- $S^{X6}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where  $\pi_1$  has smallest part  $n$ ; the parts  $m$  of  $\pi_2$  satisfy  $1 \leq m \leq n-1$ ; the parts  $m$  of  $\pi_3$  satisfying  $1 \leq m \leq 2n-1$  are odd; and the parts  $m$  of  $\pi_3$  satisfying  $2n \leq m$  are even.
- $S^{F3}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where the parts of  $\pi_1$  are even and the smallest part is  $2n$ ;  $\pi_2$  contains exactly one copy of the negative integer  $-n$ ; and the positive parts  $m$  of  $\pi_2$  are odd and satisfy  $1 \leq m \leq 2n-1$ .
- $S^{G1}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where the parts of  $\pi_1$  are even and the smallest part is  $2n$ ; the parts  $m$  of  $\pi_2$  are multiples of 4 and satisfy  $1 \leq m \leq 4n$ ; the parts  $m$  of  $\pi_3$  satisfy  $2n+1 \leq m$ ; and the odd parts of  $\pi_3$  are distinct.
- $S^{G3}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where the parts of  $\pi_1$  are even and the smallest part is  $2n$ ; the parts  $m$  of  $\pi_2$  are multiples of 4 and satisfy  $1 \leq m \leq 4n$ ; the parts  $m$  of  $\pi_3$  satisfy  $2n \leq m$ ; the odd parts of  $\pi_3$  are distinct; and the part  $2n$  appears exactly once in  $\pi_3$ .
- $S^{Y1}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3, \pi_4)$  where the parts of  $\pi_1$  are even and the smallest part is  $2n$ ; the parts  $m$  of  $\pi_2$  are multiples of 4 and satisfy  $1 \leq m \leq 4n$ ; the parts  $m$  of  $\pi_3$  satisfying  $1 \leq m \leq 2n-1$  are odd; the parts  $m$  of  $\pi_3$  satisfying  $2n+2 \leq m$  are even; the parts  $2n$  and  $2n+1$  do not appear in  $\pi_3$ ; and  $\pi_4$  consists of exactly one copy of the part  $n^2 - 2n$  (which in the case of  $n = 1$  is  $-1$ ).
- $S^{Y2}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3, \pi_4)$  where the parts of  $\pi_1$  are even and the smallest part is  $2n$ ; the parts  $m$  of  $\pi_2$  are multiples of 4 and satisfy  $1 \leq m \leq 4n$ ; the parts  $m$  of  $\pi_3$  satisfying  $1 \leq m \leq 2n-1$  are odd; the parts  $m$  of  $\pi_3$  satisfying  $2n+2 \leq m$  are even; the parts  $2n$  and  $2n+1$  do not appear in  $\pi_3$ ; and  $\pi_4$  consists of exactly one copy of the part  $n^2$ .
- $S^{Y3}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where the parts of  $\pi_1$  are even and the smallest part is  $2n$ ; the parts  $m$  of  $\pi_2$  are multiples of 4 and satisfy  $1 \leq m \leq 4n$ ; the parts  $m$  of  $\pi_3$  satisfying  $1 \leq m \leq 2n-1$  are odd; the parts  $m$  of  $\pi_3$  satisfying  $2n+2 \leq m$  are even; and the parts  $2n$  and  $2n+1$  do not appear in  $\pi_3$ .
- $S^{Y4}$  - the set of vector partitions  $(\pi_1, \pi_2, \pi_3)$  where the parts of  $\pi_1$  are even and the smallest part is  $2n$ ; the parts  $m$  of  $\pi_2$  are multiples of 4 and satisfy  $1 \leq m \leq 4n$ ; the parts  $m$  of  $\pi_3$  satisfying

$1 \leq m \leq 2n - 1$  are odd; the parts  $m$  of  $\pi_3$  satisfying  $2n \leq m$  are even; and the part  $2n$  appears exactly once in  $\pi_3$ .

- $S^{L2}$  - the set of vector partitions  $(\pi_1, \pi_2)$  where the parts of  $\pi_1$  are multiples of 4 and the smallest part is  $4n$ ; the parts  $m$  of  $\pi_2$  satisfy  $m \not\equiv 2 \pmod{4}$  and  $4n + 1 \leq m$ ; and the parts  $m$  of  $\pi_2$  that are multiples of 4 additionally satisfy  $m \leq 8n$ .

**Theorem 4.2.** *Suppose the assumptions and notation of Theorem 2.1 and let*

$$\beta'_n(q) = (q^{n+1})_\infty (q)_n^2 P_X(q) \beta_n.$$

*Suppose  $k$  is a positive integer and let  $A$  denote the set of all compositions of  $k$ , then*

$$\begin{aligned} \sum_{n=1}^{\infty} \text{spt}_k^X(n) q^n &= \sum_{(m_1, \dots, m_r) = \vec{m} \in A} \sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \dots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \\ &\quad \times \binom{f_{j_2}+m_2}{2m_2} \dots \binom{f_{j_r}+m_r}{2m_r} q^{n_1 f_1 + n_{j_2} f_{j_2} + \dots + n_{j_r} f_{j_r}} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^i}. \end{aligned}$$

*In particular for all of the Bailey pairs considered in Corollary 2.2, we have for all positive  $k$  and  $n$  that*

$$\sum_{n=1}^{\infty} \text{spt}_k^X(n) q^n = \sum_{\vec{\pi} \in S^X} \omega_k(\vec{\pi}) q^{|\vec{\pi}|}.$$

*Proof.* The proof is much the same as that of Theorem 5.6 from [21] and its generalization to Theorem 3.1 from [25]. We first note that

$$\sum_{n=1}^{\infty} \text{spt}_k^X(n) q^n = \sum_{1 \leq n_1 \leq n_2 \leq \dots \leq n_k} \frac{\beta'_{n_1}(q) q^{n_1 + n_2 + \dots + n_k}}{(q^{n_1+1})_\infty (1-q^{n_2})^2 \dots (1-q^{n_k})^2}. \quad (4.1)$$

We will use that

$$\frac{x^j}{(1-x)^{2j}} = \sum_{n=j}^{\infty} \binom{n+j-1}{2j-1} x^n, \quad \frac{x^j}{(1-x)^{2j+1}} = \sum_{n=j}^{\infty} \binom{n+j}{2j} x^n.$$

To illustrate the series rearrangements of the general proof, we first write out in full detail the case when  $k = 3$ . For  $k = 3$  we have

$$\begin{aligned} \sum_{n=1}^{\infty} \text{spt}_3^X(n) q^n &= \sum_{n_{j_3} \geq n_{j_2} \geq n_1 \geq 1} \frac{\beta'_{n_1}(q) q^{n_1 + n_{j_2} + n_{j_3}}}{(q^{n_1+1})_\infty (1-q^{n_{j_3}})^2 (1-q^{n_{j_2}})^2 (1-q^{n_1})^2} \\ &= \left( \sum_{1 \leq n_1 = n_{j_2} = n_{j_3}} + \sum_{1 \leq n_1 = n_{j_2} < n_{j_3}} + \sum_{1 \leq n_1 < n_{j_2} = n_{j_3}} + \sum_{1 \leq n_1 < n_{j_2} < n_{j_3}} \right) \frac{\beta'_{n_1}(q) q^{n_1 + n_{j_2} + n_{j_3}}}{(q^{n_1+1})_\infty (1-q^{n_{j_3}})^2 (1-q^{n_{j_2}})^2 (1-q^{n_1})^2} \\ &= \sum_{1 \leq n_1} \frac{q^{3n_1}}{(1-q^{n_1})^6} \beta'_{n_1}(q) \prod_{i > n_1} \frac{1}{1-q^i} + \sum_{1 \leq n_1 < n_{j_3}} \frac{q^{2n_1}}{(1-q^{n_1})^4} \frac{q^{n_{j_3}}}{(1-q^{n_{j_3}})^3} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \neq n_{j_3}}} \frac{1}{1-q^i} \\ &\quad + \sum_{1 \leq n_1 < n_{j_2}} \frac{q^{n_1}}{(1-q^{n_1})^2} \frac{q^{2n_{j_2}}}{(1-q^{n_{j_2}})^5} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \neq n_{j_2}}} \frac{1}{1-q^i} \\ &\quad + \sum_{1 \leq n_1 < n_{j_2} < n_{j_3}} \frac{q^{n_1}}{(1-q^{n_1})^2} \frac{q^{n_{j_2}}}{(1-q^{n_{j_2}})^3} \frac{q^{n_{j_3}}}{(1-q^{n_{j_3}})^3} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \neq n_{j_2}, n_{j_3}}} \frac{1}{1-q^i} \\ &= \sum_{1 \leq n_1} \sum_{f_1=3}^{\infty} \binom{f_1+3-1}{6-1} q^{n_1 f_1} \beta'_{n_1}(q) \prod_{i > n_1} \frac{1}{1-q^i} \\ &\quad + \sum_{1 \leq n_1 < n_{j_3}} \sum_{f_1=2}^{\infty} \binom{f_1+2-1}{4-1} q^{n_1 f_1} \sum_{f_{j_3}=1}^{\infty} \binom{f_{j_3}+1}{2} q^{n_{j_3} f_{j_3}} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \neq n_{j_3}}} \frac{1}{1-q^i} \end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq n_1 < n_{j_2}} \sum_{f_1=1}^{\infty} \binom{f_1+1-1}{2-1} q^{n_1 f_1} \sum_{f_{j_2}=2}^{\infty} \binom{f_{j_2}+2}{4} q^{n_{j_2} f_{j_2}} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \neq n_{j_2}}} \frac{1}{1-q^i} \\
& + \sum_{1 \leq n_1 < n_{j_2} < n_{j_3}} \sum_{f_1=1}^{\infty} \binom{f_1+1-1}{2-1} q^{n_1 f_1} \sum_{f_{j_2}=1}^{\infty} \binom{f_{j_2}+1}{2} q^{n_{j_2} f_{j_2}} \sum_{f_{j_3}=1}^{\infty} \binom{f_{j_3}+1}{2} q^{n_{j_3} f_{j_3}} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \neq n_{j_2}, n_{j_3}}} \frac{1}{1-q^i}.
\end{aligned}$$

Now the set of compositions of 3 is  $A = \{(3), (2, 1), (1, 2), (1, 1, 1)\}$ , and so we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \text{spt}_3^X(n) q^n &= \sum_{(m_1, \dots, m_r) = \vec{m} \in A} \sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \dots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \\
&\times \binom{f_{j_2}+m_2}{2m_2} \dots \binom{f_{j_r}+m_r}{2m_r} q^{n_1 f_1 + n_{j_2} f_{j_2} + \dots + n_{j_r} f_{j_r}} \beta'(q) \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^i}.
\end{aligned}$$

For general  $k$  we take the expression for  $\sum_{n=1}^{\infty} \text{spt}_k^X(n) q^n$  in (4.1) and split the sum into  $2^{k-1}$  sums by turning the index bounds into  $<$  or  $=$ , each of which corresponds to a composition of  $k$ . In particular we recall that the  $2^{k-1}$  compositions of  $k$  can be obtained by taking a list of  $k$  ones and between each one we put either a plus or a comma. Given the index bounds  $n_1 \leq n_2 \leq \dots \leq n_k$ , we make a choice of each  $\leq$  being “=” or “<”; we associate to this the composition  $1^+ 1^+ \dots 1^+$  where in order we choose “+” when we chose “=” and choose “,” when we chose “<”.

If we let  $A$  be the set of all compositions of  $k$ , with the manipulations discussed above, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \text{spt}_k^X(n) q^n &= \sum_{(m_1, \dots, m_r) = \vec{m} \in A} \sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \dots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \\
&\times \binom{f_{j_2}+m_2}{2m_2} \dots \binom{f_{j_r}+m_r}{2m_r} q^{n_1 f_1 + n_{j_2} f_{j_2} + \dots + n_{j_r} f_{j_r}} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^i}.
\end{aligned}$$

This we recognize as the generating function for vector partitions  $\vec{\pi} = (\pi_1, \dots, \pi_r)$  counted according to the weight  $\omega_k(\vec{\pi})$  where  $\beta'_{n_1}(q)$  determines the types of partitions in  $(\pi_2, \dots, \pi_r)$ .  $\square$

We note that taking  $k = 1$  means that for  $\text{spt}_1^X(n)$  we simply count the number of appearances of the smallest part in  $\pi_1$ , and many of these functions have been studied elsewhere. In particular, those  $\text{spt}_1^X(n)$  which possess simple linear congruences were studied by Garvan and the second author in [23, 27, 26];  $\text{spt}_1^{C1}(n)$  was studied by Andrews, Dixit, and Yee in [9]; and  $\text{spt}_1^{C1}(n)$ ,  $\text{spt}_1^{C5}(n)$ , and  $\text{spt}_1^{J1}(n)$  were studied by Patkowski in [30, 31].

To demonstrate these weighted counts, we compute a table of values for  $\text{spt}_k^X(n)$  with  $X = A1, B2$ ,  $k = 1, 2, 3$ , and  $n = 4, 5$ . We note that the first three weights are given by

$$\begin{aligned}
\omega_1(\pi) &= f_1^1(\pi), \\
\omega_2(\pi) &= \binom{f_1^1(\pi)+1}{3} + f_1^1(\pi) \sum_{2 \leq j} \binom{f_j^1(\pi)+1}{2}, \\
\omega_3(\pi) &= \binom{f_1^1(\pi)+2}{5} + \binom{f_1^1(\pi)+1}{3} \sum_{2 \leq j} \binom{f_j^1(\pi)+1}{2} + f_1^1(\pi) \sum_{2 \leq j} \binom{f_j+2}{4} \\
&\quad + f_1^1(\pi) \sum_{2 \leq j < k} \binom{f_j^1(\pi)+1}{2} \binom{f_k^1(\pi)+1}{2}.
\end{aligned}$$



TABLE 2.  $A_1$  Partitions of 4

	$f_1^1$	$f_2^1$	$f_3^1$	$\omega_1$	$\omega_2$	$\omega_3$
$(4, \emptyset)$	1	0	0	1	0	0
$(3+1, \emptyset)$	1	1	0	1	1	0
$(2+2, \emptyset)$	2	0	0	2	1	0
$(2+1+1, \emptyset)$	2	1	0	2	3	1
$(1+1+1+1, \emptyset)$	4	0	0	4	10	6
$(1+1, 2)$	2	0	0	2	1	0
Total				12	16	7

TABLE 3.  $A_1$  Partitions of 5

	$f_1^1$	$f_2^1$	$f_3^1$	$\omega_1$	$\omega_2$	$\omega_3$
$(5, \emptyset)$	1	0	0	1	0	0
$(4+1, \emptyset)$	1	1	0	1	1	0
$(3+2, \emptyset)$	1	1	0	1	1	0
$(3+1+1, \emptyset)$	2	1	0	2	3	1
$(2+2+1, \emptyset)$	1	2	0	1	3	1
$(2+1+1+1, \emptyset)$	3	1	0	3	7	5
$(1+1+1+1+1, \emptyset)$	5	0	0	5	20	21
$(2, 3)$	1	0	0	1	0	0
$(2+1, 2)$	1	1	0	1	1	0
$(1+1+1, 2)$	3	0	0	3	4	1
$(1, 2+2)$	1	0	0	1	0	0
Total				20	40	29

TABLE 4.  $B_2$  Partitions of 4

	$f_1^1$	$f_2^1$	$f_3^1$	$\omega_1$	$\omega_2$	$\omega_3$
$(2, 2)$	1	0	0	1	0	0
$(2+1, 1)$	1	1	0	1	1	0
$(1+1+1, 1)$	3	0	0	3	4	1
Total				5	5	1

TABLE 5.  $B_2$  Partitions of 5

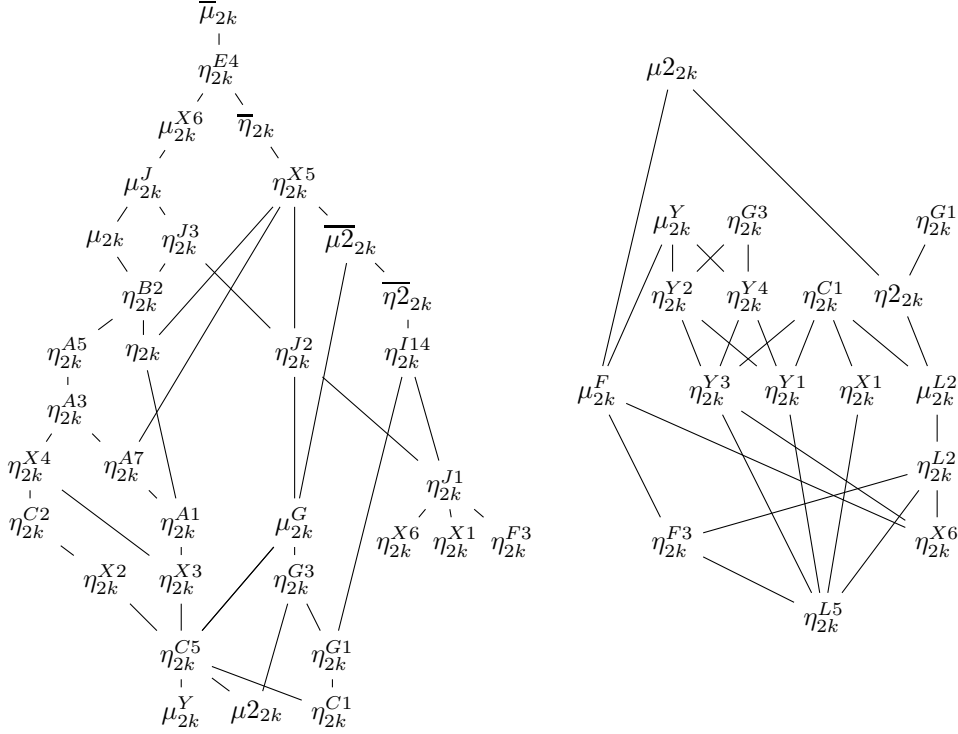
	$f_1^1$	$f_2^1$	$f_3^1$	$\omega_1$	$\omega_2$	$\omega_3$
$(3+1, 1)$	1	1	0	1	1	0
$(2+1+1, 1)$	2	1	0	2	3	1
$(1+1+1+1, 1)$	4	0	0	4	10	6
Total				7	14	7

## 5. CONJECTURES AND CONCLUDING REMARKS

As demonstrated, Bailey's Lemma is well suited to give inequalities between the moments of rank-like Lambert series and corresponding crank functions, as well as supplying the combinatorial interpretation of the difference of the symmetrized moments. In our work, we focused our attention to the  $a = 1$  Bailey pairs of Slater [33, 34] with  $\alpha_n = \alpha_{-n}$ . We have also seen that it is possible to work with Bailey pairs from other sources that satisfy the same conditions. However it is not true that all Bailey pairs, even from Slater's lists, satisfy  $\alpha_n = \alpha_{-n}$ . We leave it open to the interested reader to see to what extent one can develop similar identities that lead to similar results.

We have studied these new rank moments as far as possible while handling them in generality. However, we expect each function to possess interesting properties of its own. In particular, based on how the four prototypical examples from [21, 25] behave, we expect each ordinary rank moment generating function to correspond to a quasi-mock modular form, each ordinary crank moment to correspond to a quasi-modular form, and potentially an equation exists relating the certain partial derivatives of these rank and crank functions. Using these automorphic properties one could hope to derive asymptotic formulas for the moments, such as was done in [19, 20].

As we have already seen, a large number of inequalities exist between the various moments. We conjecture the following diagram gives all of the inequalities. Here a downward path from  $A_{2k}$  to  $B_{2k}$  indicates  $A_{2k}(n) \geq B_{2k}(n)$  for all positive  $k$  and  $n$ . We have split the diagram into two pieces, to decrease the height and handle the large number of crossings. Besides the rank and crank moments defined in this article, we also include the overpartition rank  $\overline{\eta}_{2k}$ , the the overpartition M2-rank  $\overline{\eta}_{2k}^2$ , the overpartiton M2-crank  $\overline{\mu}_{2k}^2$ , the M2-rank for partitions without repeated odd parts  $\eta_{2k}$ , and the M2-crank for partitions without repeated odd parts  $\mu_{2k}$  from [25]. It is likely some of these inequalities can be proved using the identities of this article, but to get the full picture one will need additional techniques. These inequalities have been verified for  $1 \leq k \leq 10$  and  $1 \leq n < 1000$ .



Additionally based on numerical evidence, it would appear that some of the higher order spt functions satisfy a number of congruences. While a few of these may follow by elementary means, to approach these likely one should start with the automorphic properties of the moments mentioned above, as this is the method that worked for the original examples. We conjecture the following congruences,

$$\begin{aligned}
0 &\equiv \text{spt}_3^{B2}(4n+3) \equiv \text{spt}_2^{E4}(31n) \equiv \text{spt}_2^{E4}(41n) \equiv \text{spt}_2^{E4}(47n) \equiv \text{spt}_2^{E4}(16n+1) \equiv \text{spt}_2^{E4}(32n+2) \\
&\equiv \text{spt}_2^{E4}(8n+7) \equiv \text{spt}_2^{E4}(49n+7) \equiv \text{spt}_2^{E4}(49n+14) \equiv \text{spt}_2^{E4}(18n+15) \equiv \text{spt}_2^{E4}(24n+17) \\
&\equiv \text{spt}_2^{E4}(40n+17) \equiv \text{spt}_2^{E4}(36n+21) \equiv \text{spt}_2^{E4}(45n+21) \equiv \text{spt}_2^{E4}(49n+21) \equiv \text{spt}_2^{E4}(32n+28) \\
&\equiv \text{spt}_2^{E4}(49n+28) \equiv \text{spt}_2^{E4}(36n+30) \equiv \text{spt}_2^{E4}(40n+33) \equiv \text{spt}_2^{E4}(45n+33) \equiv \text{spt}_2^{E4}(48n+34) \\
&\equiv \text{spt}_2^{E4}(49n+35) \equiv \text{spt}_2^{E4}(45n+39) \equiv \text{spt}_2^{E4}(45n+42) \equiv \text{spt}_2^{E4}(49n+42) \equiv \text{spt}_3^{E4}(16n+3) \\
&\equiv \text{spt}_3^{E4}(32n+10) \equiv \text{spt}_3^{E4}(16n+13) \equiv \text{spt}_3^{E4}(49n+21) \equiv \text{spt}_3^{E4}(32n+22) \equiv \text{spt}_3^{E4}(49n+35) \\
&\equiv \text{spt}_3^{E4}(49n+42) \equiv \text{spt}_4^{E4}(31n) \equiv \text{spt}_4^{E4}(47n) \equiv \text{spt}_4^{E4}(32n+1) \equiv \text{spt}_4^{E4}(16n+7) \equiv \text{spt}_4^{E4}(32n+14) \\
&\equiv \text{spt}_4^{E4}(48n+17) \equiv \text{spt}_5^{E4}(32n+11) \equiv \text{spt}_5^{E4}(32n+13) \equiv \text{spt}_6^{E4}(31n) \equiv \text{spt}_6^{E4}(32n+17) \\
&\equiv \text{spt}_6^{E4}(32n+23) \equiv \text{spt}_7^{E4}(32n+29) \equiv \text{spt}_8^{E4}(31n) \equiv \text{spt}_8^{E4}(32n+7) \equiv \text{spt}_3^{X3}(4n+2) \equiv \text{spt}_3^{X4}(44n+28) \\
&\equiv \text{spt}_2^{X5}(48n+1) \equiv \text{spt}_2^{X5}(24n+7) \equiv \text{spt}_2^{X5}(48n+14) \equiv \text{spt}_2^{X5}(24n+17) \equiv \text{spt}_2^{X5}(24n+23) \\
&\equiv \text{spt}_2^{X5}(48n+34) \equiv \text{spt}_2^{X5}(48n+46) \equiv \text{spt}_3^{X5}(48n+11) \equiv \text{spt}_3^{X5}(48n+13) \equiv \text{spt}_3^{X5}(48n+29) \\
&\equiv \text{spt}_3^{X5}(48n+43) \equiv \text{spt}_4^{X5}(48n+7) \equiv \text{spt}_4^{X5}(48n+17) \equiv \text{spt}_4^{X5}(32n+23) \equiv \text{spt}_4^{X5}(48n+23) \\
&\equiv \text{spt}_2^{X6}(46n+3) \equiv \text{spt}_2^{X6}(49n+15) \equiv \text{spt}_2^{X6}(49n+29) \equiv \text{spt}_2^{X6}(41n+36) \equiv \text{spt}_2^{X6}(49n+36) \\
&\equiv \text{spt}_3^{X6}(4n) \equiv \text{spt}_3^{X6}(18n+2) \equiv \text{spt}_3^{X6}(18n+14) \equiv \text{spt}_3^{X6}(49n+15) \equiv \text{spt}_3^{X6}(49n+29) \\
&\equiv \text{spt}_3^{X6}(49n+36) \equiv \text{spt}_6^{X6}(8n) \equiv \text{spt}_6^{X6}(8n+7) \equiv \text{spt}_6^{X6}(41n+36) \equiv \text{spt}_2^{F3}(4n+1) \equiv \text{spt}_3^{F3}(4n) \\
&\equiv \text{spt}_4^{F3}(8n+1) \equiv \text{spt}_5^{F3}(8n+4) \equiv \text{spt}_6^{F3}(8n+5) \equiv \text{spt}_8^{F3}(16n+1) \equiv \text{spt}_{10}^{F3}(16n+5) \pmod{2}, \\
0 &\equiv \text{spt}_2^{A3}(9n) \equiv \text{spt}_4^{B2}(3n) \equiv \text{spt}_3^{E4}(27n+15) \equiv \text{spt}_4^{E4}(27n+6) \equiv \text{spt}_2^{X5}(9n) \equiv \text{spt}_2^{X5}(24n+11) \\
&\equiv \text{spt}_2^{X5}(32n+12) \equiv \text{spt}_8^{X5}(27n) \equiv \text{spt}_2^{F3}(6n+1) \equiv \text{spt}_3^{L2}(27n+26) \pmod{3}, \\
0 &\equiv \text{spt}_4^{L5}(44n+28) \equiv \text{spt}_2^{E4}(16n+14) \equiv \text{spt}_2^{E4}(36n+33) \pmod{4},
\end{aligned}$$

$$\begin{aligned}
0 &\equiv \text{spt}_2^{A1}(5n) \equiv \text{spt}_2^{A1}(5n+1) \equiv \text{spt}_4^{A1}(25n+24) \equiv \text{spt}_5^{A1}(25n+24) \equiv \text{spt}_2^{A3}(5n+1) \equiv \text{spt}_2^{A3}(5n+2) \\
&\equiv \text{spt}_2^{A3}(5n+4) \equiv \text{spt}_2^{A5}(5n) \equiv \text{spt}_2^{A5}(5n+4) \equiv \text{spt}_3^{A5}(25n+9) \equiv \text{spt}_3^{A5}(25n+14) \equiv \text{spt}_2^{A7}(5n+1) \\
&\equiv \text{spt}_2^{A7}(5n+4) \equiv \text{spt}_3^{A7}(25n+24) \equiv \text{spt}_2^{B2}(5n+1) \equiv \text{spt}_2^{B2}(5n+2) \equiv \text{spt}_2^{B2}(5n+4) \equiv \text{spt}_3^{B2}(25n+1) \\
&\equiv \text{spt}_3^{B2}(25n+9) \equiv \text{spt}_4^{B2}(25n+2) \equiv \text{spt}_4^{B2}(25n+4) \equiv \text{spt}_8^{B2}(25n+12) \equiv \text{spt}_2^{C5}(5n) \equiv \text{spt}_2^{C5}(5n+1) \\
&\equiv \text{spt}_2^{C5}(5n+4) \equiv \text{spt}_3^{C5}(25n+24) \equiv \text{spt}_4^{C5}(25n+3) \equiv \text{spt}_4^{C5}(25n+5) \equiv \text{spt}_4^{C5}(25n+24) \\
&\equiv \text{spt}_5^{C5}(25n+4) \equiv \text{spt}_5^{C5}(25n+24) \equiv \text{spt}_6^{C5}(25n+5) \equiv \text{spt}_6^{C5}(25n+10) \equiv \text{spt}_2^{E4}(5n) \equiv \text{spt}_2^{E4}(5n+2) \\
&\equiv \text{spt}_3^{F3}(25n+3) \equiv \text{spt}_3^{F3}(25n+23) \equiv \text{spt}_{5k}^{Y1}(10n+3) \equiv \text{spt}_5^{Y1}(25n+8) \equiv \text{spt}_{10}^{Y1}(25n+8) \\
&\equiv \text{spt}_{5k}^{Y2}(10n+9) \pmod{5}, \\
0 &\equiv \text{spt}_2^{A1}(49n+12) \equiv \text{spt}_3^{A1}(49n+47) \equiv \text{spt}_2^{A3}(49n+19) \equiv \text{spt}_2^{A5}(7n+1) \equiv \text{spt}_3^{A5}(7n) \equiv \text{spt}_3^{A5}(7n+1) \\
&\equiv \text{spt}_3^{A5}(7n+3) \equiv \text{spt}_3^{A5}(7n+5) \equiv \text{spt}_6^{A5}(7n+5) \equiv \text{spt}_2^{A7}(7n) \equiv \text{spt}_2^{A7}(7n+1) \equiv \text{spt}_3^{A7}(7n) \\
&\equiv \text{spt}_3^{A7}(7n+1) \equiv \text{spt}_3^{A7}(7n+2) \equiv \text{spt}_3^{A7}(7n+4) \equiv \text{spt}_5^{A7}(49n+47) \equiv \text{spt}_6^{A7}(49n+47) \\
&\equiv \text{spt}_2^{B2}(7n+1) \equiv \text{spt}_2^{B2}(7n+5) \equiv \text{spt}_3^{B2}(7n) \equiv \text{spt}_3^{B2}(7n+1) \equiv \text{spt}_3^{B2}(7n+3) \equiv \text{spt}_3^{B2}(7n+5) \\
&\equiv \text{spt}_4^{B2}(49n+1) \equiv \text{spt}_5^{B2}(49n+33) \equiv \text{spt}_6^{B2}(7n+5) \pmod{7}, \\
0 &\equiv \text{spt}_2^{E4}(32n+30) \pmod{8}, \\
0 &\equiv \text{spt}_5^{X5}(27n) \equiv \text{spt}_5^{X5}(27n+18) \pmod{9}, \\
0 &\equiv \text{spt}_2^{B2}(11n+1) \pmod{11}, \\
0 &\equiv \text{spt}_3^{A5}(25n+24) \equiv \text{spt}_2^{B2}(25n+14) \equiv \text{spt}_2^{C5}(25n+24) \equiv \text{spt}_2^{F3}(25n+23) \pmod{25}, \\
0 &\equiv \text{spt}_3^{B2}(49n+26) \pmod{49}.
\end{aligned}$$

Lastly, there is also the concept of positive moments, where  $m$  ranges over just the positive integers, rather than all of  $\mathbb{Z}$ , that is to say,

$$\begin{aligned}
N_k^+(n) &= \sum_{m=1}^{\infty} m^k N(m, n), & \eta_k^+(n) &= \sum_{m=1}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m, n), \\
M_k^+(n) &= \sum_{m=1}^{\infty} m^k M(m, n), & \mu_k^+(n) &= \sum_{m=1}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m, n).
\end{aligned}$$

The advantage to these positive moments is that while  $N_{2k}(n) = 2N_{2k}^+(n)$  and  $M_{2k}(n) = 2M_{2k}^+(n)$ , it is no longer the case that the odd moments are zero. It is true that  $M_{2k+1}^+(n) > N_{2k+1}^+(n)$ , and there have been several studies of the positive moments corresponding to the rank and crank of ordinary partitions as well as overpartitions, see [1, 8, 18, 28, 35]. As such we should expect that analogous results and inequalities hold for the moments of this article, however our methods do not directly apply and it is not clear if one can handle positive moments in the generality we have managed for the original moments.

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